

# *Colourful linear programming and simplicial depth*

**Antoine Deza** (McMaster)

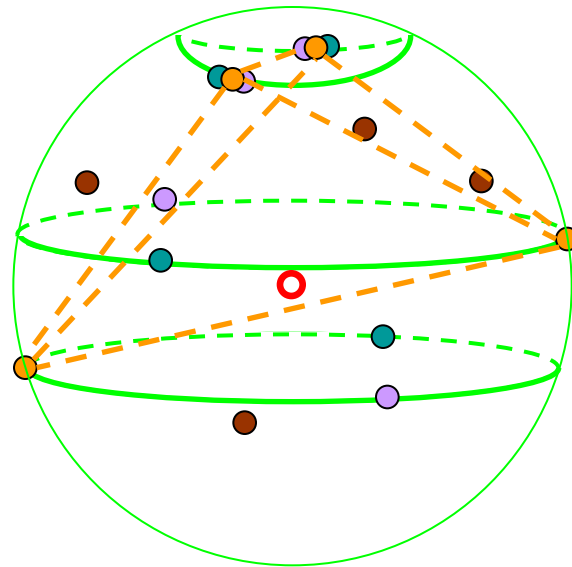
based on joint works with

**Frédéric Meunier** (ENPC)

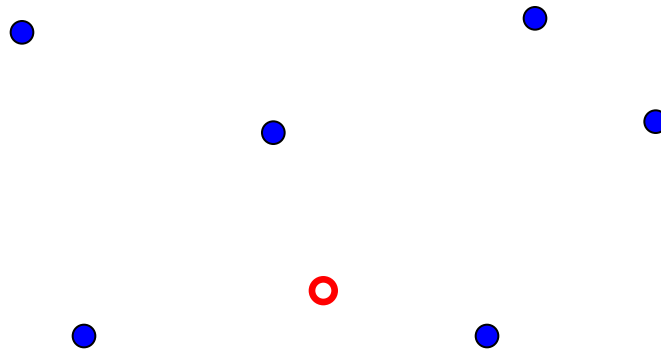
**Tamon Stephen** (Simon Fraser)

**Pauline Sarrabezolles** (ENPC)

**Feng Xie** (Microsoft)



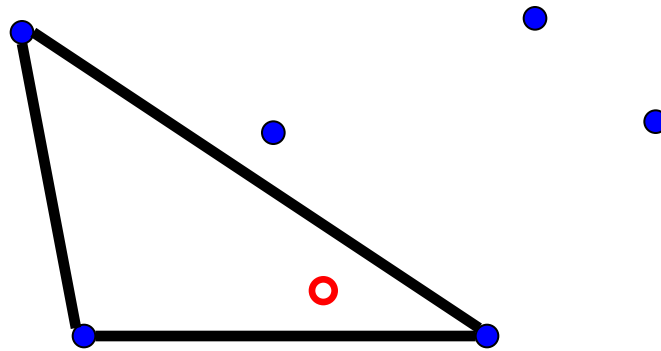
# *Carathéodory Theorem*



Given a set  $S$  of  $n$  points in dimension  $d$ , then there exists an open simplex generated by points in  $S$  containing  $p$

$S, p$  general position

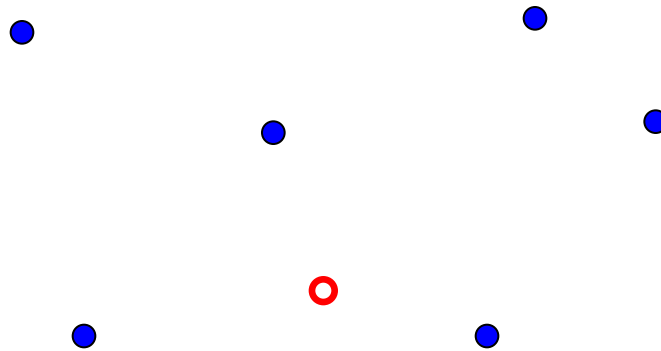
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# *Simplicial Depth*

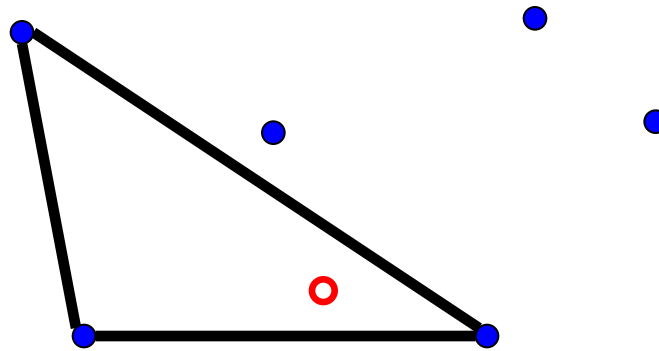


Given a set  $S$  of  $n$  points in dimension  $d$ , the *simplicial depth* of  $p$  is the number of open simplices generated by points in  $S$  containing  $p$  [Liu 1990]

$S, p$  general position

# *Simplicial Depth*

$$\text{depth}_S(p) = 1$$

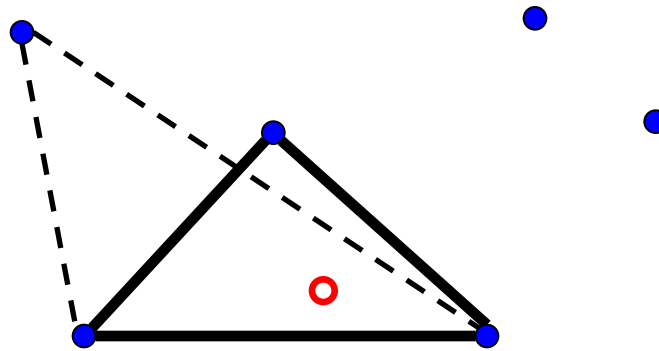


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$S, p$  general position

# *Simplicial Depth*

$$\text{depth}_S(p) = 2$$

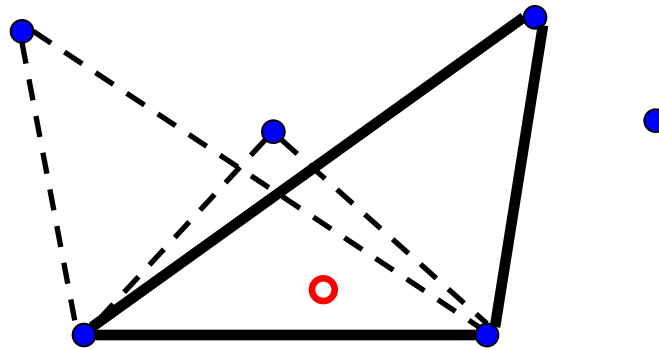


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$S, p$  general position

# *Simplicial Depth*

$$\text{depth}_S(p) = 3$$

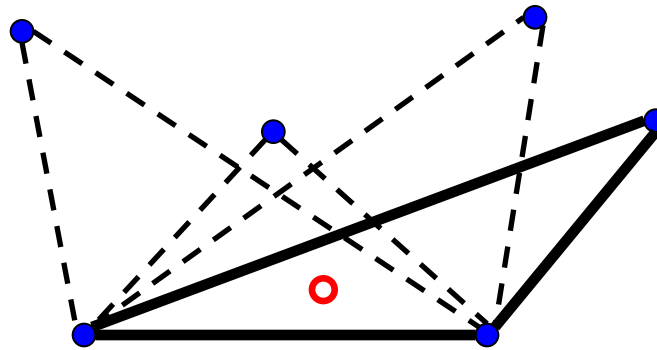


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$S, p$  general position

# *Simplicial Depth*

$$\text{depth}_S(p) = 4$$



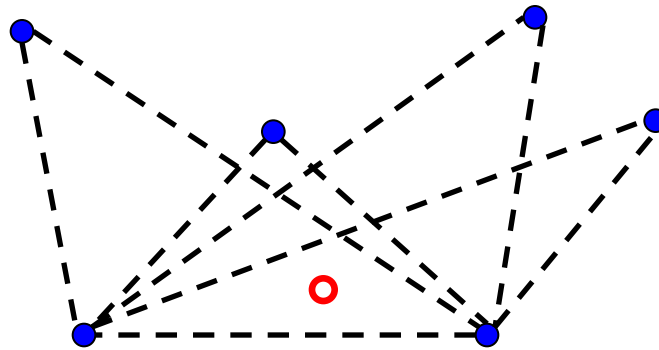
Given a set  $S$  of  $n$  points in dimension  $d$ , the *simplicial depth* of  $p$  is the number of open simplices generated by points in  $S$  containing  $p$  [Liu 1990]

$S, p$  general position



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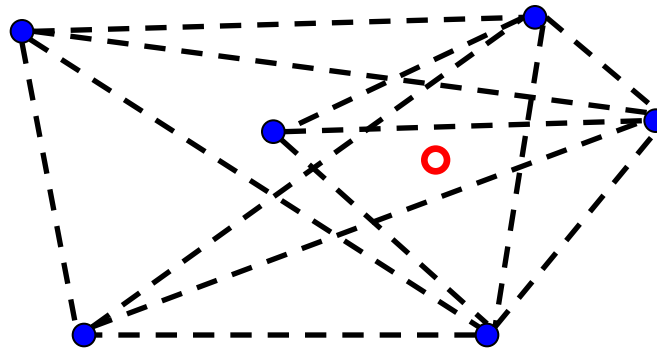


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$S, p$  general position

# *Simplicial Depth*

$$\text{depth}_S(p) = 9$$



Given a set  $S$  of  $n$  points in dimension  $d$ , the *simplicial depth* of  $p$  is the number of open simplices generated by points in  $S$  containing  $p$  [Liu 1990]

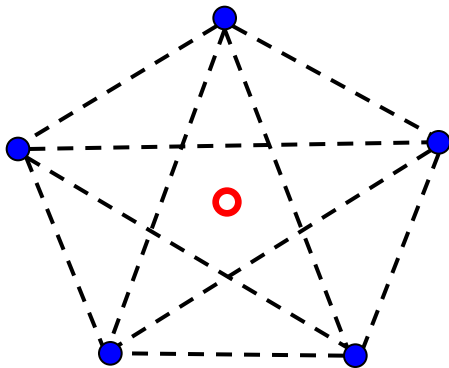
$S, p$  general position

# Deepest Point in Dimension 2

Deepest point bounds in dimension 2 [Kárteszi 1955],  
[Boros, Füredi 1984]

$$\frac{n^3}{27} + O(n^2) \leq \max_p \text{depth}_S(p) \leq \frac{n^3}{24} + O(n^2)$$

and [Bukh, Matoušek, Nivasch 2010]



$$\text{depth}_S(p) = 5$$

$S, p$  general position

# *Deepest Point in Dimension $d$*

Deepest point bounds in dimension  $d$  [Bárány 1982]

$$\frac{1}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d) \leq \max_p \text{depth}_S(p) \leq \frac{1}{2^d (d+1)!} n^{d+1} + O(n^d)$$

$S, p$  general position

# *Deepest Point in Dimension $d$*

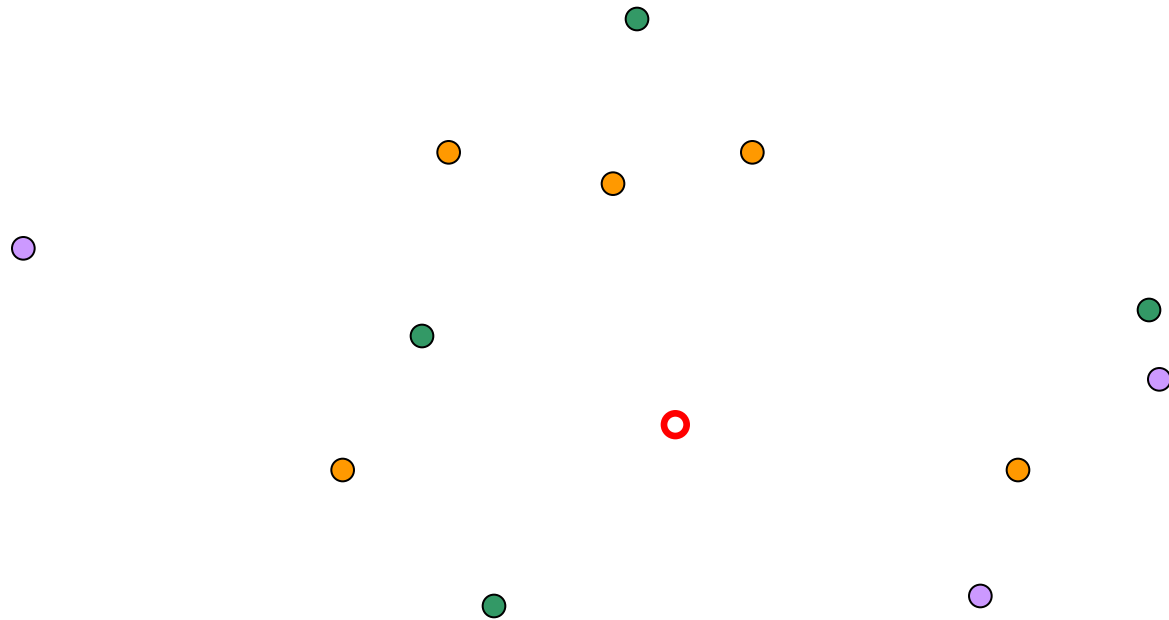
Deepest point bounds in dimension  $d$  [Bárány 1982]

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- tight upper bound
- lower bound uses Colourful Carathéodory theorem

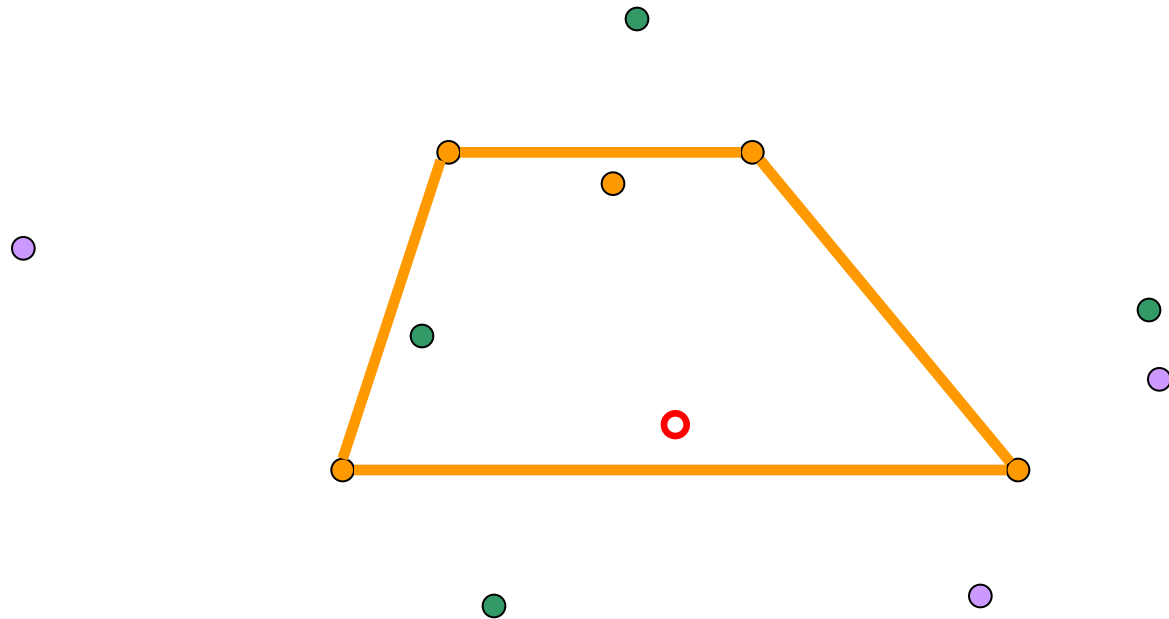
... breakthrough [Gromov 2010] & further improvements

# *Colourful Carathéodory Theorem*



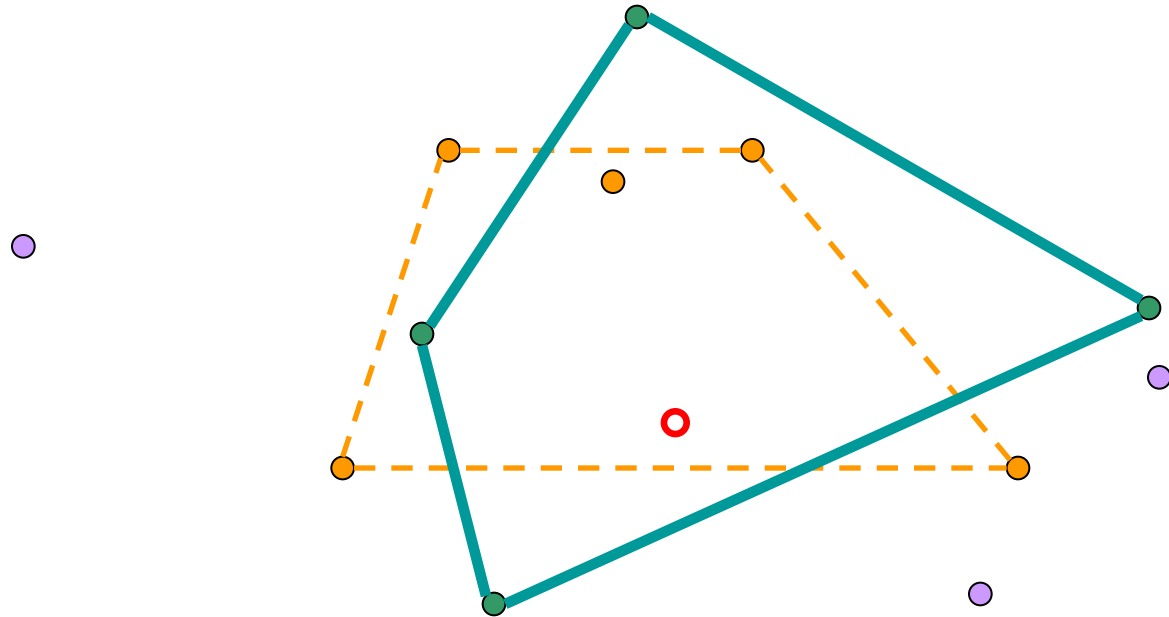
Given colourful set  $\mathbf{S} = S_1 \cup S_2 \dots \cup S_{d+1}$  in dimension  $d$ , and  $p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$ , there exists a colourful simplex containing  $p$  [Bárány 1982]

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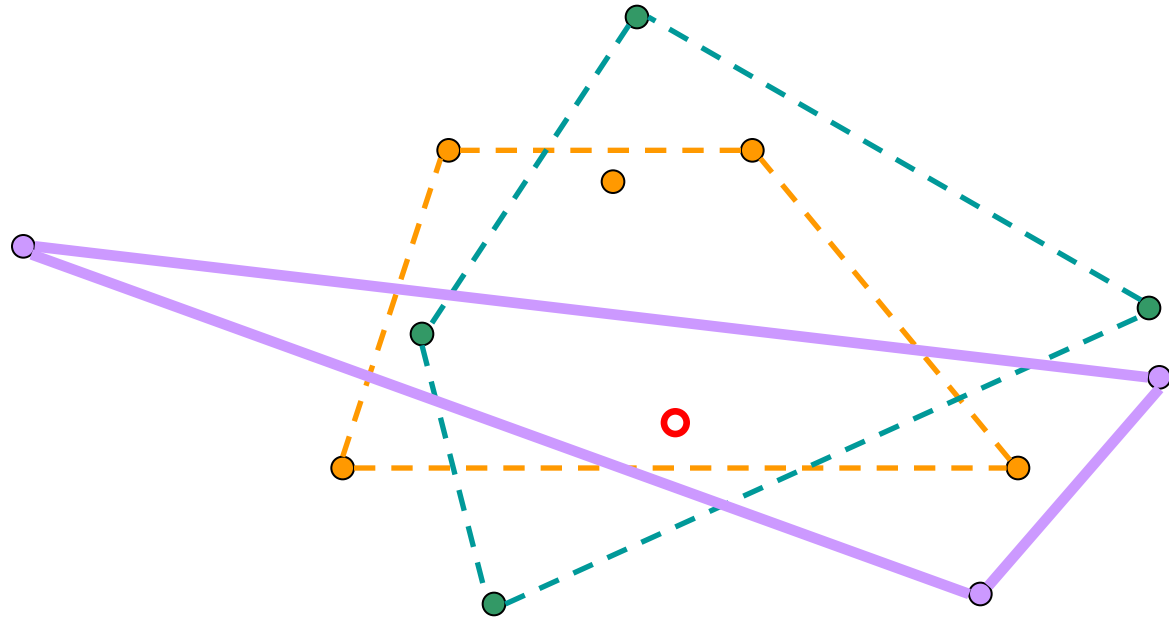
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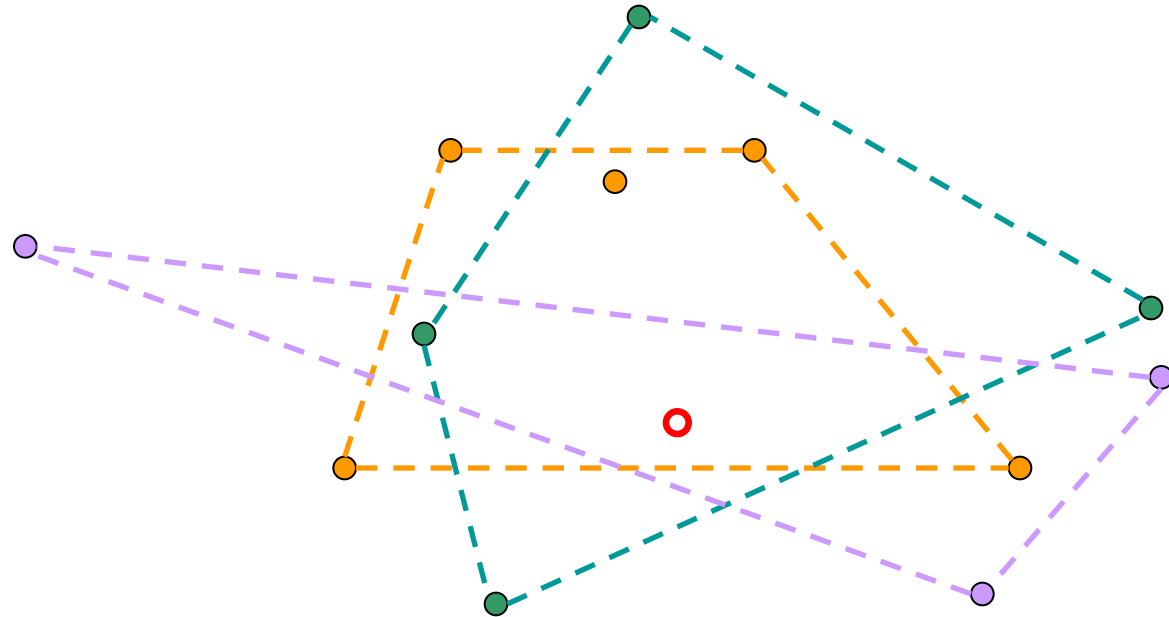


# *Colourful Carathéodory Theorem*



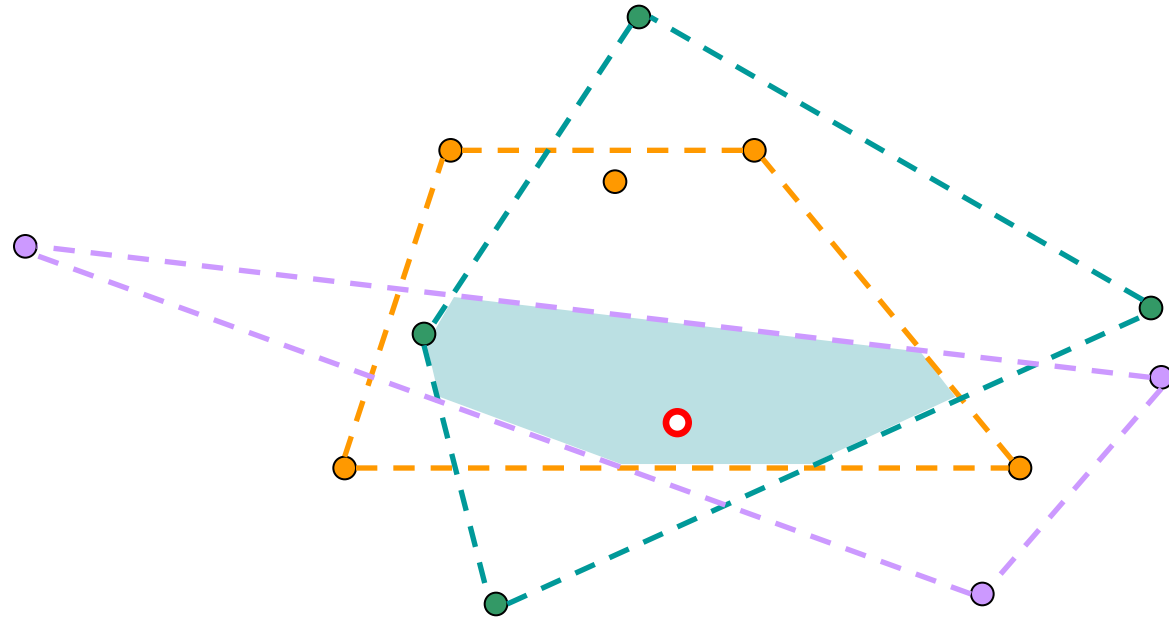
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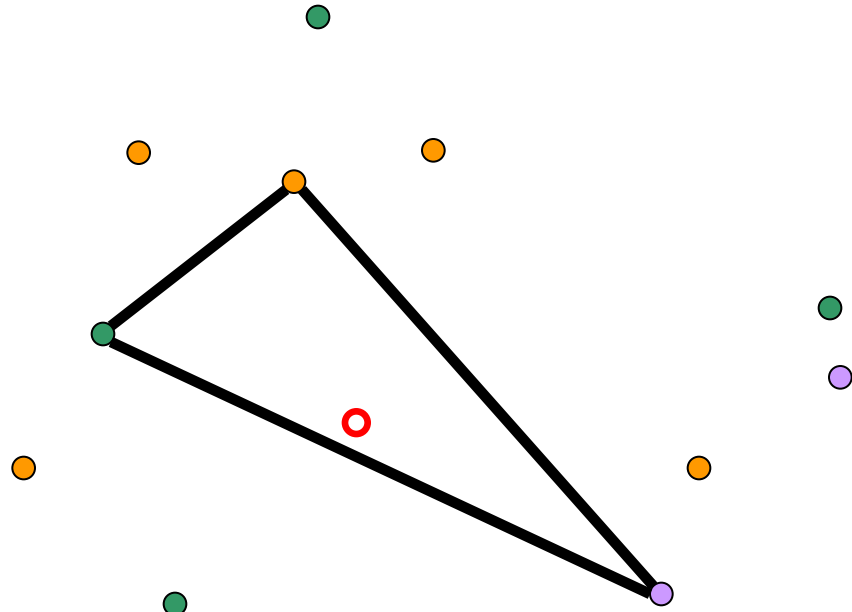
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# Colourful Simplicial Depth

$$\text{depth}_{\mathbf{S}}(p) = 1$$



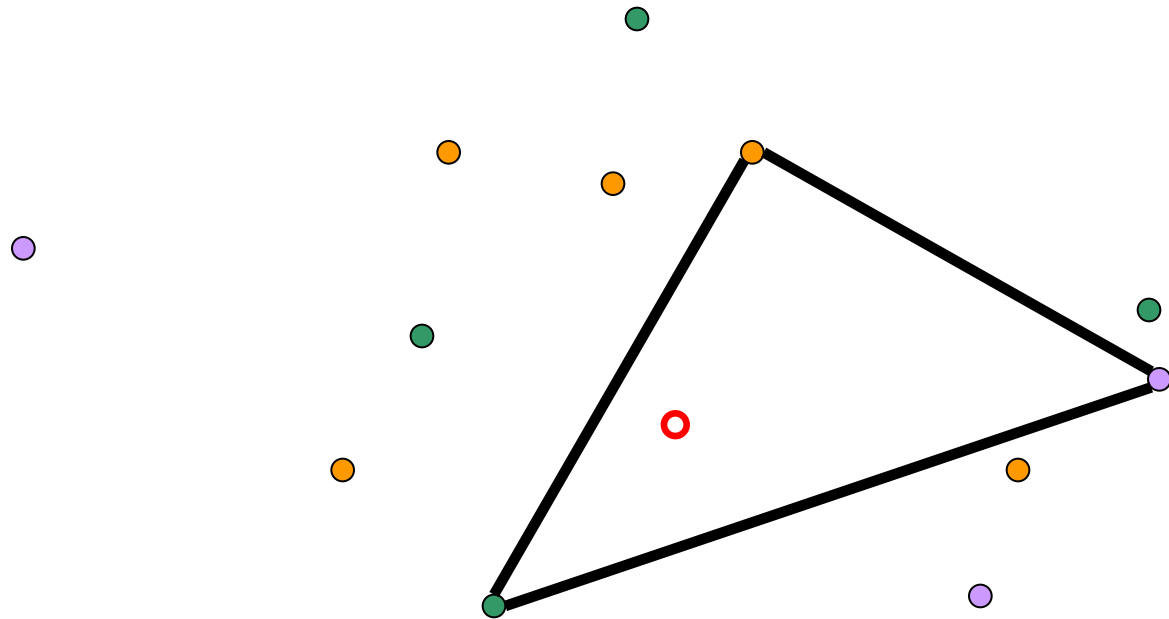
Given colourful set  $\mathbf{S} = S_1 \cup S_2 \dots \cup S_{d+1}$  in dimension  $d$ , the colourful simplicial depth of  $p$  is the number of open colourful simplexes generated by points in  $\mathbf{S}$  containing  $p$

$\mathbf{S}, p$  general position

$$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$$

# Colourful Simplicial Depth

$$\text{depth}_{\mathcal{S}}(p) = 2$$



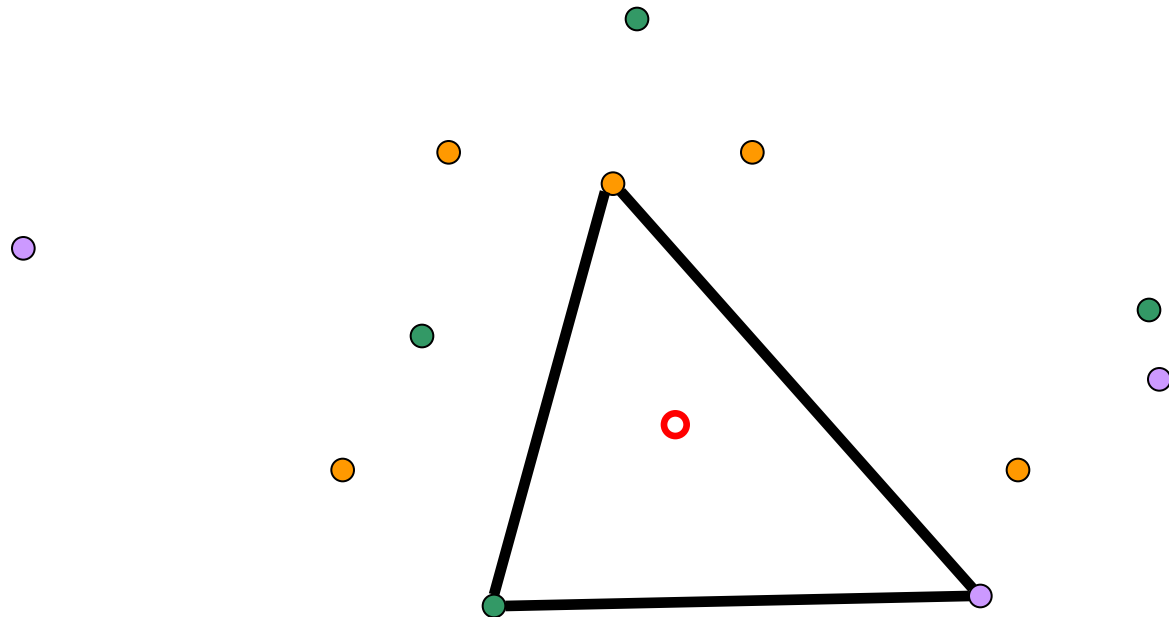
Given colourful set  $\mathcal{S} = \mathcal{S}_1 \cup \mathcal{S}_2 \dots \cup \mathcal{S}_{d+1}$  in dimension  $d$ , the colourful simplicial depth of  $p$  is the number of open colourful simplexes generated by points in  $\mathcal{S}$  containing  $p$

$\mathcal{S}, p$  general position

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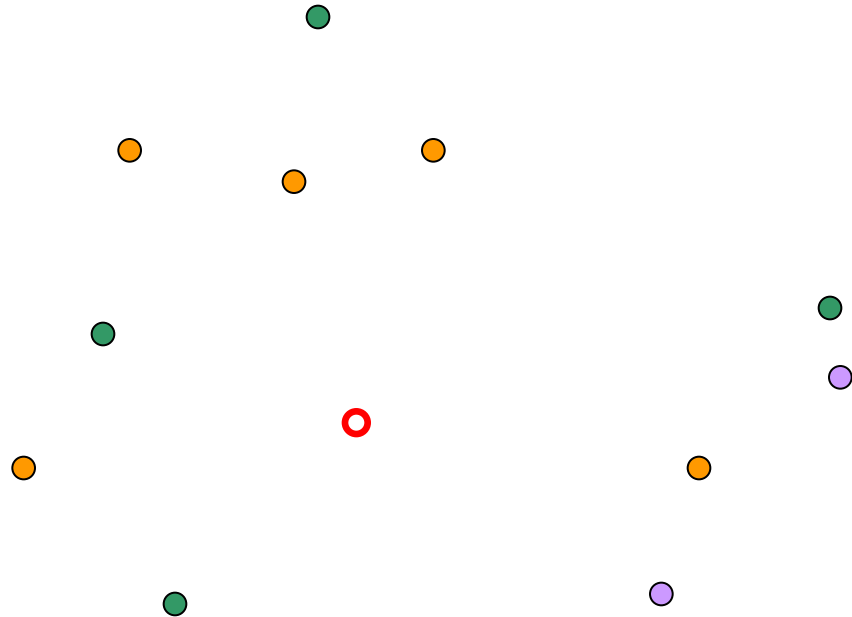
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# Colourful Simplicial Depth

$$\text{depth}_{\mathbf{S}}(p) = 16$$



Given colourful set  $\mathbf{S} = S_1 \cup S_2 \dots \cup S_{d+1}$  in dimension  $d$ , the colourful simplicial depth of  $p$  is the number of open colourful simplices generated by points in  $\mathbf{S}$  containing  $p$

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# *Deepest Point in Dimension $d$*

Deepest point bounds in dimension  $d$  [Bárány 1982]

$$\frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d) \leq \max_p \text{depth}_S(p) \leq \frac{1}{2^d (d+1)!} n^{d+1} + O(n^d)$$

with  $\mu(d) = \min_{S,p} \text{depth}_S(p)$

[Bárány 1982]:  $\mu(d) \geq 1$

...breakthrough [Gromov 2010] & further improvements

$S, p$  general position



# *Deepest Point in Dimension $d$*

$$\max_p \text{depth}_S(p) \geq c_d \binom{n}{d+1}$$

[Bárány 1982]  $c_d \geq \frac{d+1}{(d+1)^{(d+1)}}$

[Wagner 2003]  $c_d \geq \frac{d^2+1}{(d+1)^{(d+1)}}$

[Gromov 2010]  $c_d \geq \frac{2d}{(d+1)!(d+1)}$

simpler proofs: [Karazev 2012], [Matoušek, Wagner 2012]

$d=2$ : [Boros, Füredi 1984], [Bukh 2006]

$d=3$ : [Král, Mach, Sereni 2012]

$S, p$  general position

# Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for  $\mu(d) = \min_{S, p} \text{depth}_S(p)$
- Computational approaches for  $\mu(d)$  for small  $d$ .
- Obtain an efficient algorithm to find a colourful simplex :  
*Colourful Linear Programming Feasibility* problem

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

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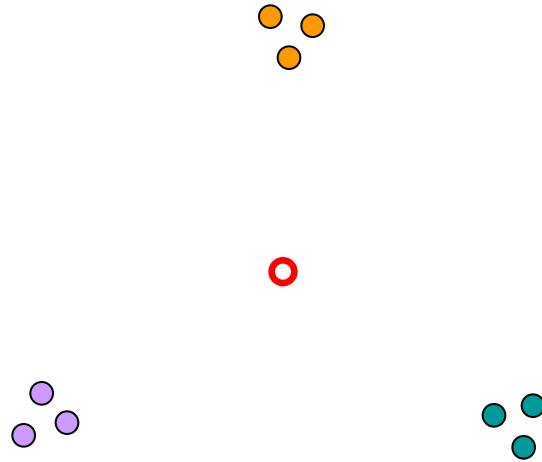
# Colourful Carathéodory Theorems

[Bárány 1982] Given colourful set  $\mathbf{S} = S_1 \cup S_2 \dots \cup S_{d+1}$  in dimension  $d$ , and  $p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$ , then there exists a colourful simplex containing  $p$

[Holmsen, Pach, Tverberg 2008] and [Arocha, Bárány, Bracho, Fabila, Montejano 2009] Given colourful set  $\mathbf{S} = S_1 \cup S_2 \dots \cup S_{d+1}$  in dimension  $d$ , and  $p \in \text{conv}(S_i \cup S_j)$  for  $1 \leq i < j \leq d+1$ , then there exists a colourful simplex containing  $p$

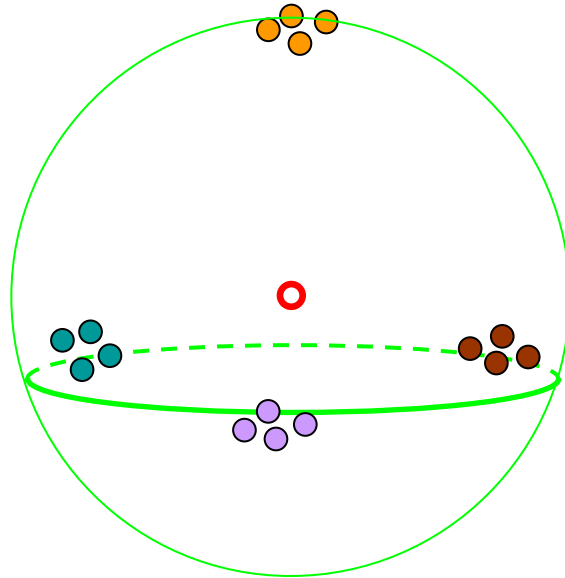
[Meunier, D. 2013] Given colourful set  $\mathbf{S} = S_1 \cup S_2 \dots \cup S_{d+1}$  in dimension  $d$ , if for  $1 \leq i < j \leq d+1$  there exists  $k \neq i, k \neq j$ , such that for all  $x_k \in S_k$  the ray  $[x_k p)$  intersects  $\text{conv}(S_i \cup S_j)$  in a point distinct from  $x_k$ , then there exists a colourful simplex containing  $p$

# Colourful Carathéodory Theorems



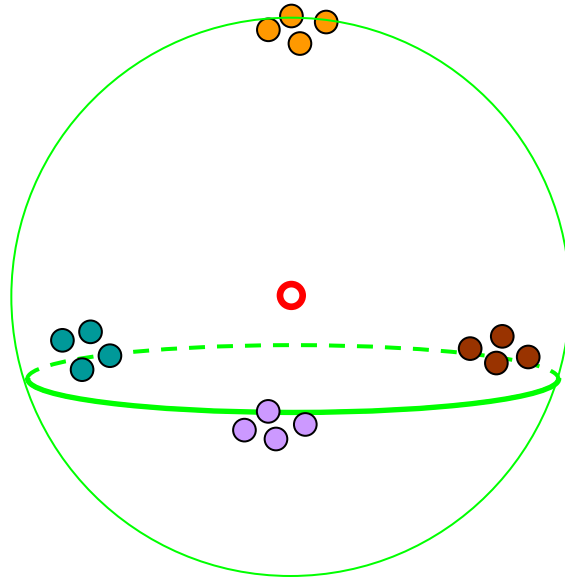
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[Meunier, D. 2013] Given colourful set  $\mathbf{S} = S_1 \cup S_2 \dots \cup S_{d+1}$  in dimension  $d$ , if for  $i \neq j$  the open half-space containing  $p$  and defined by an  $i$ -facet of a colourful simplex intersects  $S_i \cup S_j$ , then there exists a colourful simplex containing  $p$

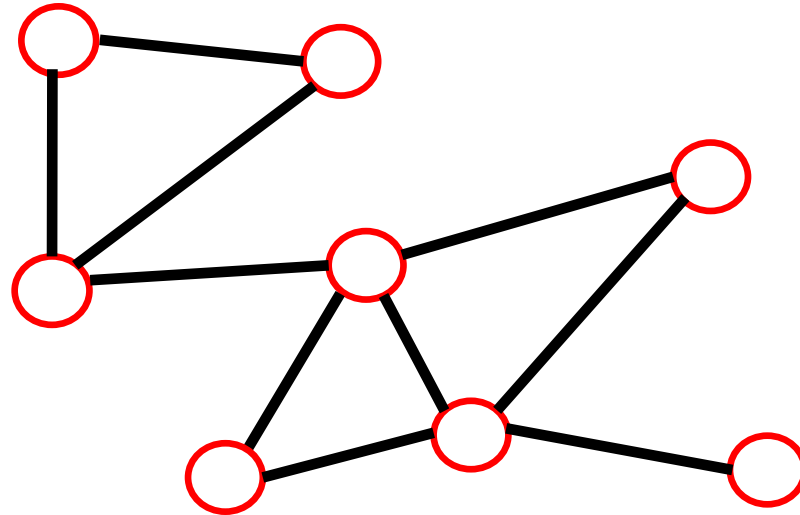
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❖ *further generalization in dimension 2*

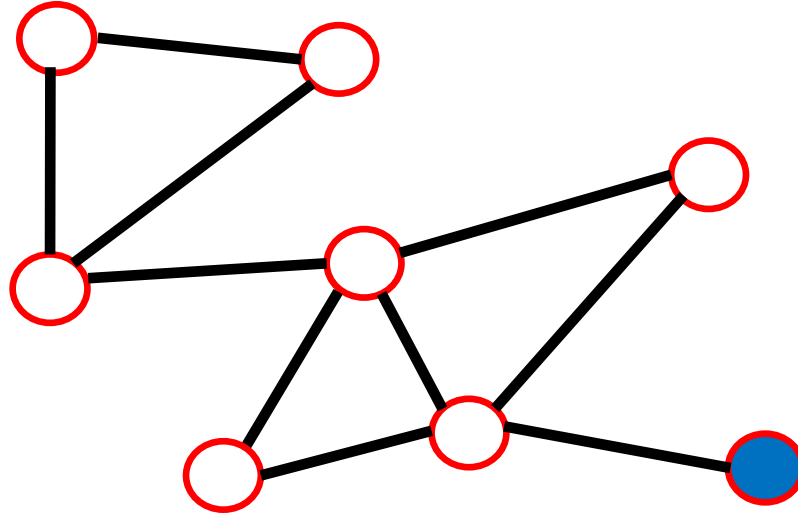
# *Given One, Get Another One*



In a graph, if there is a vertex with an odd degree...

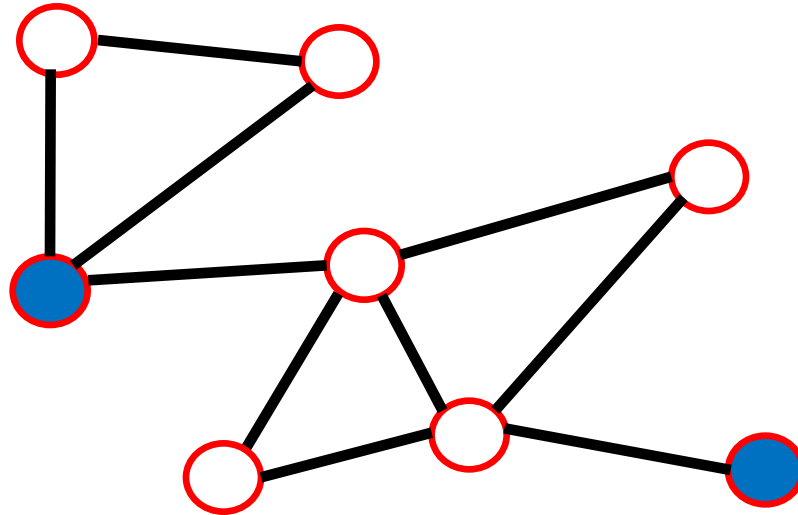


# *Given One, Get Another One*



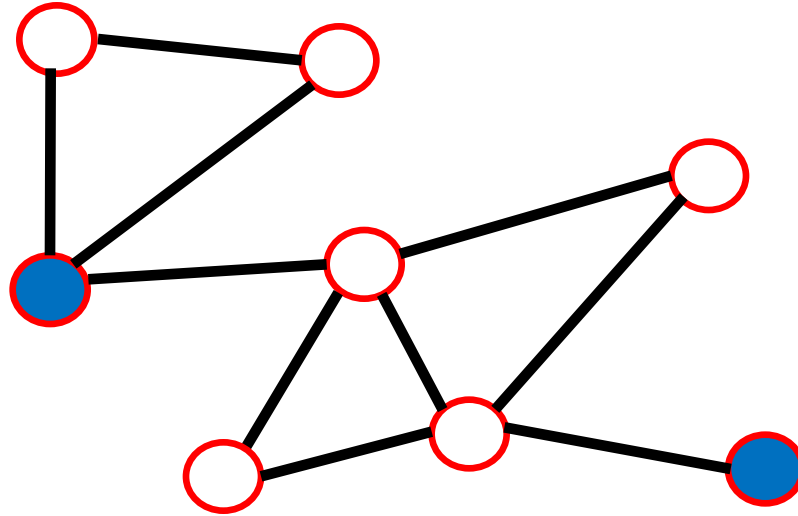
In a graph, if there is a vertex with an **odd degree**...

# *Given One, Get Another One*



In a graph, if there is a vertex with an **odd degree**... then there is **another one**

# *Given One, Get Another One*



In a graph, if there is a vertex with an **odd degree**... then there is **another one**

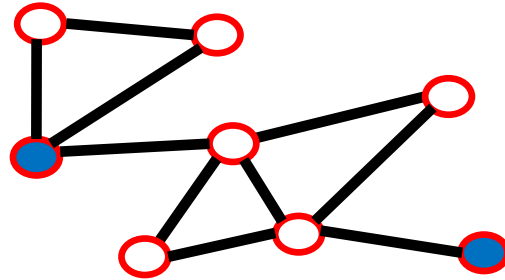
➤ *Duoid / oik* room partitioning Todd 1974, Edmonds 2009]

(*Exchange algorithm*: generalization of Lemke-Howson for finding a Nash equilibrium for a 2 players game)

➤ *Polynomial Parity Argument* PPA(D) [Papadimitriou 1994]

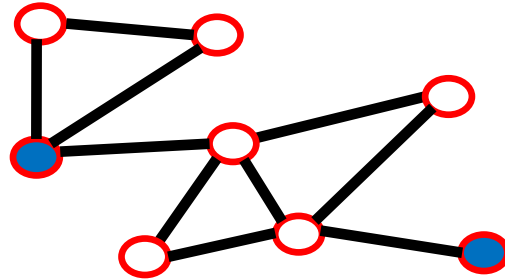
(Hamiltonian circuit in a cubic graph, Borsuk-Ulam, ...)

# *Given One, Get Another One*



[Meunier, D. 2013] Given colourful set  $\mathbf{S} = S_1 \cup S_2 \dots \cup S_{d+1}$  in dimension  $d$  with  $|S_i|=2$ , if there is a colourful simplex containing  $p$  then there is another one

# *Given One, Get Another One*



[Meunier, D. 2013] Any condition implying the existence of a colourful **simplex** containing  $p$  actually implies that the number of such simplices is at least  $d+1$

$$\mathbf{S}, p \text{ general position and } |S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$$

# Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- **Improve lower bound** for  $\mu(d) = \min_{S, p} \text{depth}_S(p)$
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$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

[Bárány 1982]

$$1 \leq \mu(d)$$

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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$$d + 1 \leq \mu(d)$$

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[Bárány 1982]

$$d + 1 \leq \mu(d)$$

[D., Huang, Stephen, Terlaky 2006]

$$2d \leq \mu(d) \leq d^2 + 1$$

$\mu(d)$  even for odd  $d$

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

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[D., Huang, Stephen, Terlaky 2006]

$$2d \leq \mu(d) \leq d^2 + 1$$

$\mu(d)$  even for odd  $d$

[Bárány, Matoušek 2007]

$$\max\left(3d, \frac{d^2 + d}{5}\right) \leq \mu(d) \quad \text{for } d \geq 3$$

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

[Bárány 1982]

$$d + 1 \leq \mu(d)$$

[D., Huang, Stephen, Terlaky 2006]

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$$3d \leq \mu(d) \quad \text{for } d \geq 3$$

[Stephen, Thomas 2008]

$$\left\lfloor \frac{(d+2)^2}{4} \right\rfloor \leq \mu(d) \quad \text{for } d \geq 8$$

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d) \quad \begin{array}{l} p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1}) \\ S, p \text{ general position and } |S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1 \end{array}$$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

[Bárány 1982]  $d + 1 \leq \mu(d)$

[D., Huang, Stephen, Terlaky 2006]  $2d \leq \mu(d) \leq d^2 + 1$

[Bárány, Matoušek 2007]  $\max(3d, \frac{d^2 + d}{5}) \leq \mu(d)$  for  $d \geq 3$

[Stephen, Thomas 2008]  $\left\lfloor \frac{(d+2)^2}{4} \right\rfloor \leq \mu(d)$  for  $d \geq 8$

[D., Stephen, Xie 2011]  $\left\lfloor \frac{(d+1)^2}{2} \right\rfloor \leq \mu(d)$  for  $d \geq 4$

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

$$\mu(1) = 2 \quad \mu(2) = 5 \quad \mu(3) = 10$$

$$\left\lceil \frac{(d+1)^2}{2} \right\rceil \leq \mu(d) \leq d^2 + 1 \quad \text{for } d \geq 4$$

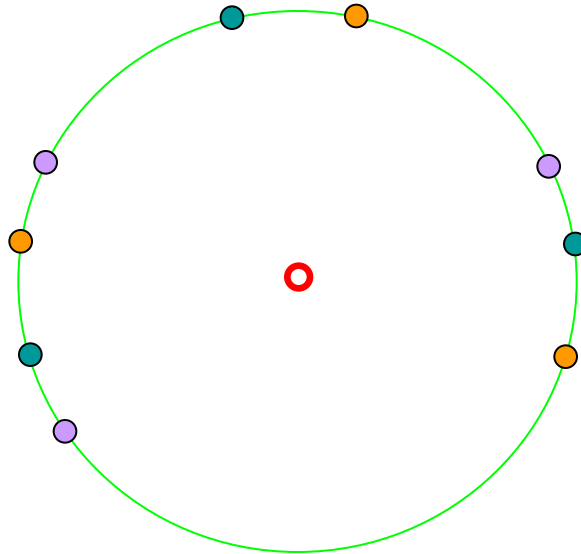
$\mu(d)$  even for odd  $d$

conjecture:  $\mu(d) = d^2 + 1$

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d) \quad \begin{array}{l} p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1}) \\ S, p \text{ general position and } |S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1 \end{array}$$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

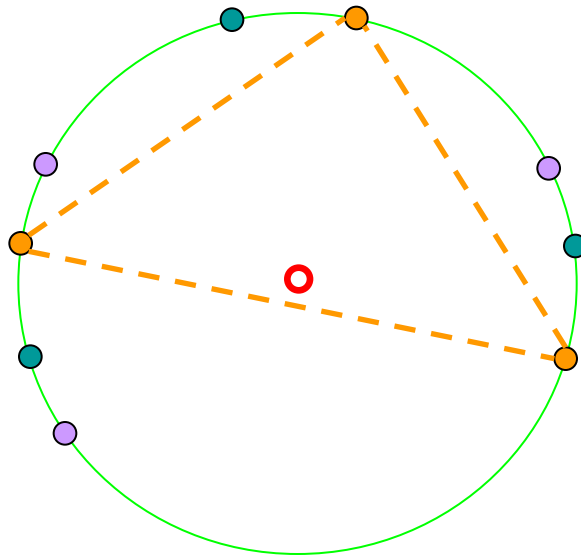


$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$



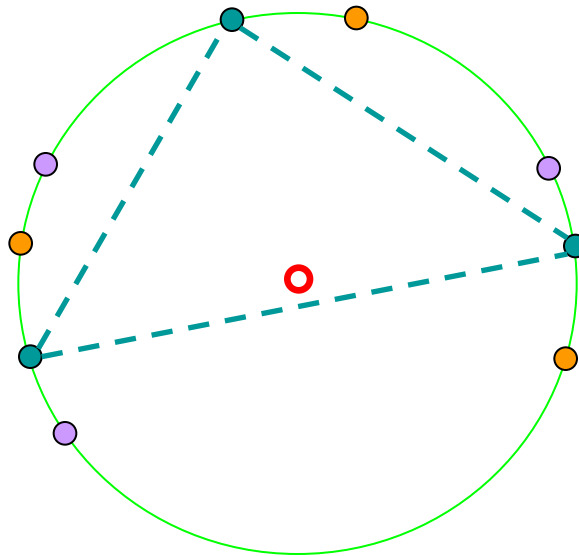
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$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$



# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

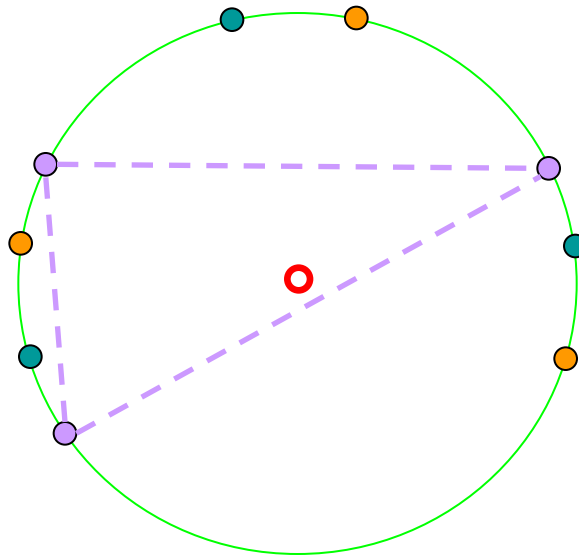


$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$



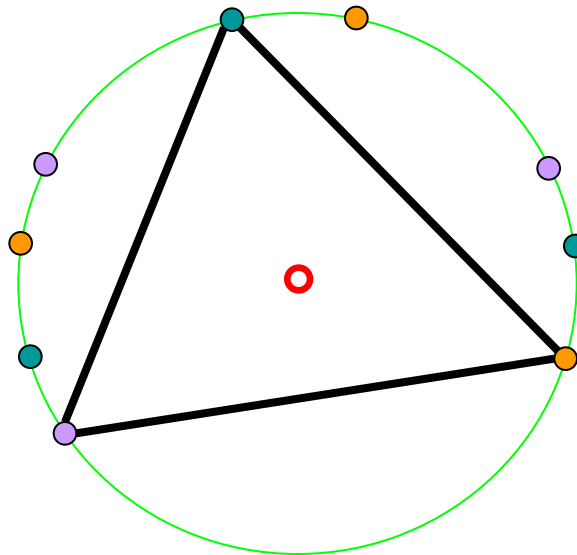
$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

$$\text{depth}_S(p) = 1$$



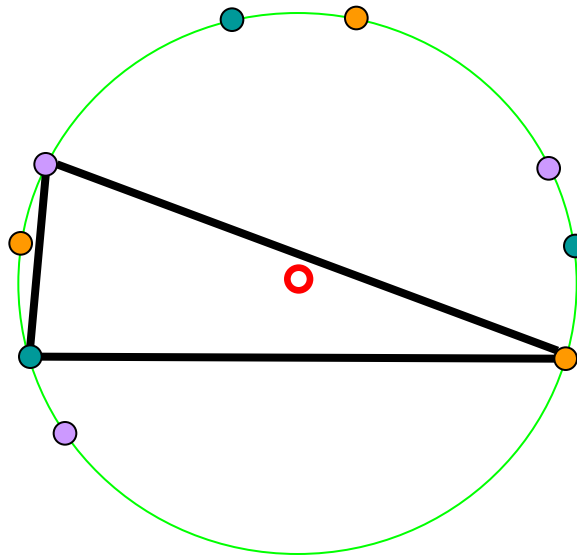
$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

$$\text{depth}_S(p) = 2$$



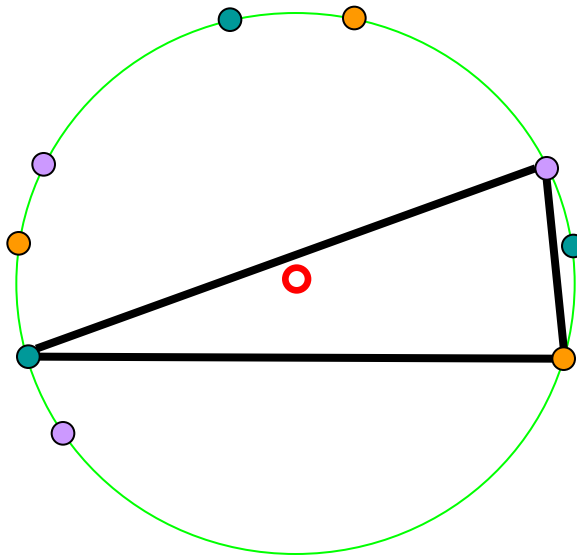
$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

$$\text{depth}_S(p) = 3$$



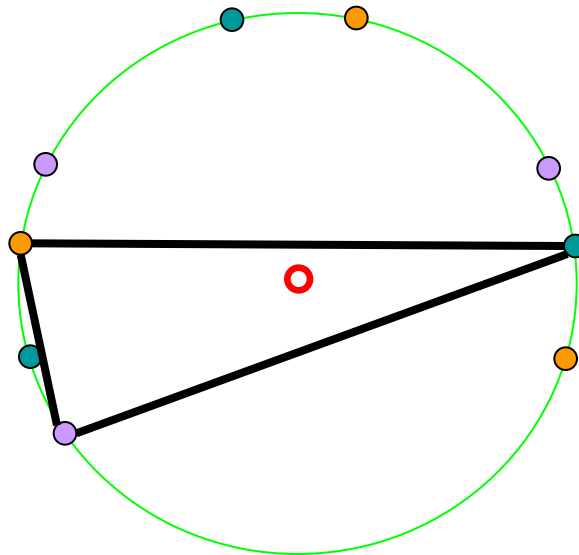
$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

$$\text{depth}_S(p) = 4$$



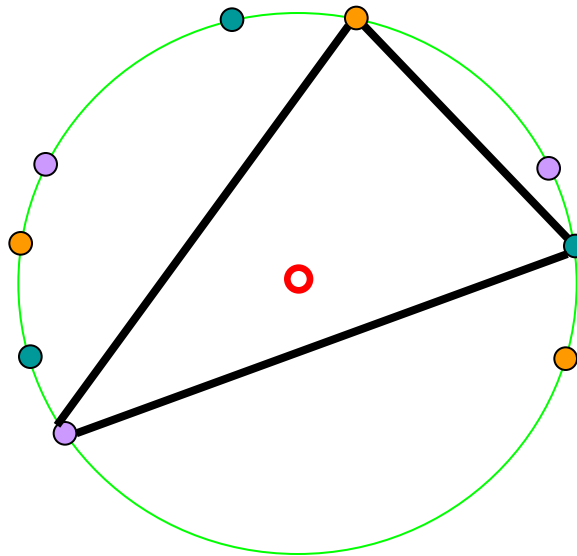
$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

$$\text{depth}_S(p) = 5$$

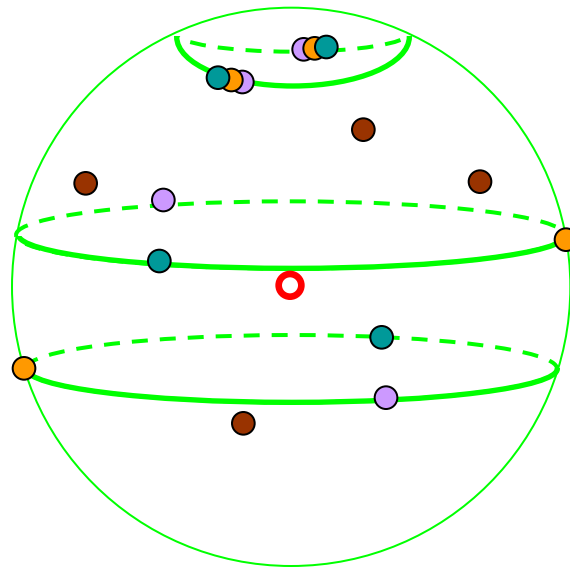


$$\mu(2) = 5$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$



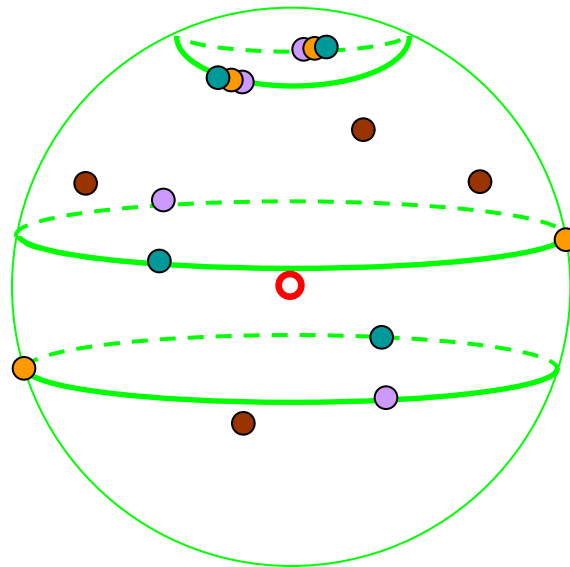
$$3d \leq \mu(d) \leq d^2 + 1$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$



# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

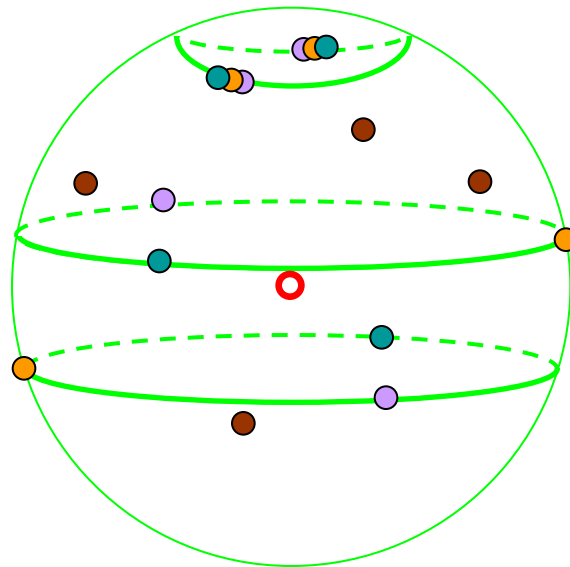


$$9 \leq \mu(3) \leq 10$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$



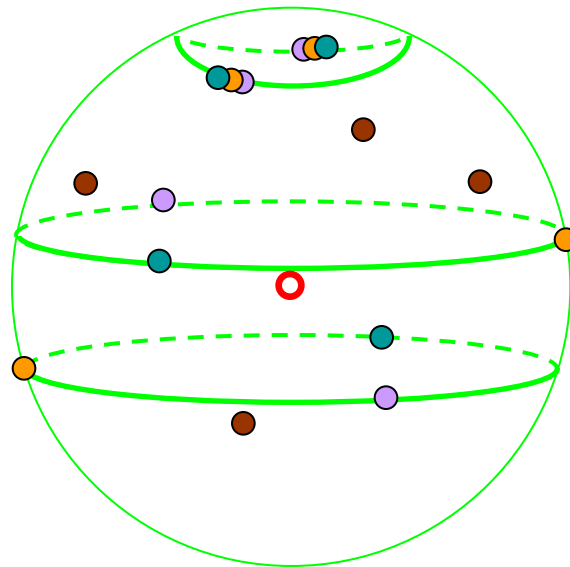
$$9 \leq \mu(3) \leq 10$$

$\mu(d)$  even for odd  $d$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
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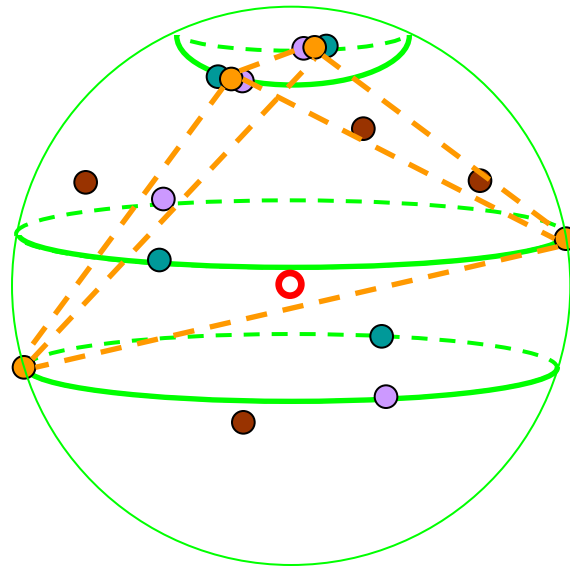


$$\mu(3) = 10$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
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# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

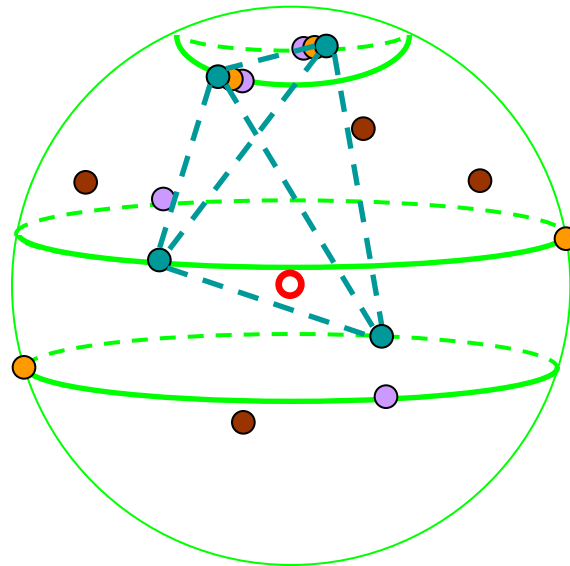


$$\mu(3) = 10$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
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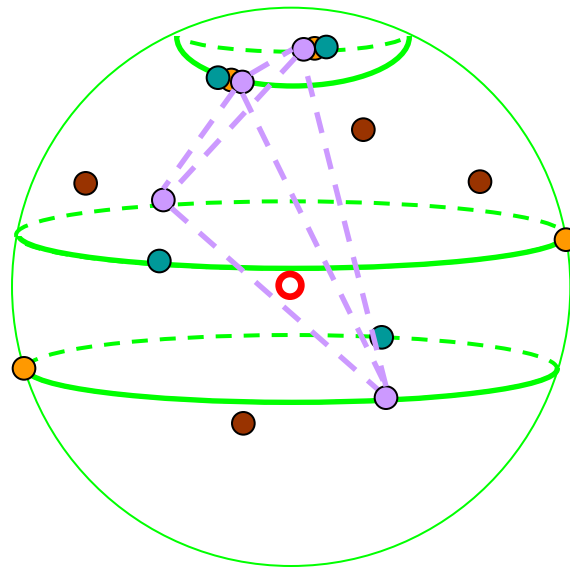


$$\mu(3) = 10$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

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$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

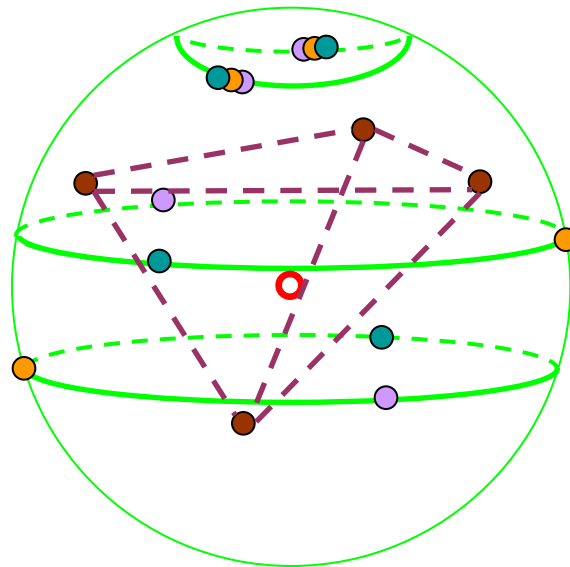


$$\mu(3) = 10$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

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$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$



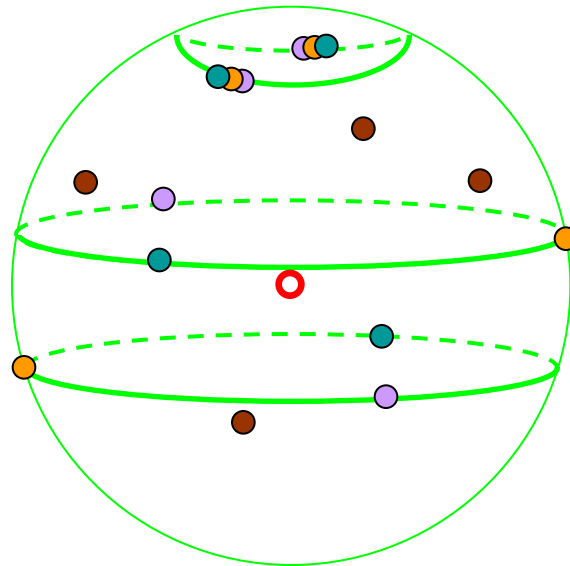
$$\mu(3) = 10$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
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$$\mu(d) = \min_{S, p} \text{depth}_S(p)$$

$$\text{depth}_S(p) = 10$$



$$\mu(3) = 10$$

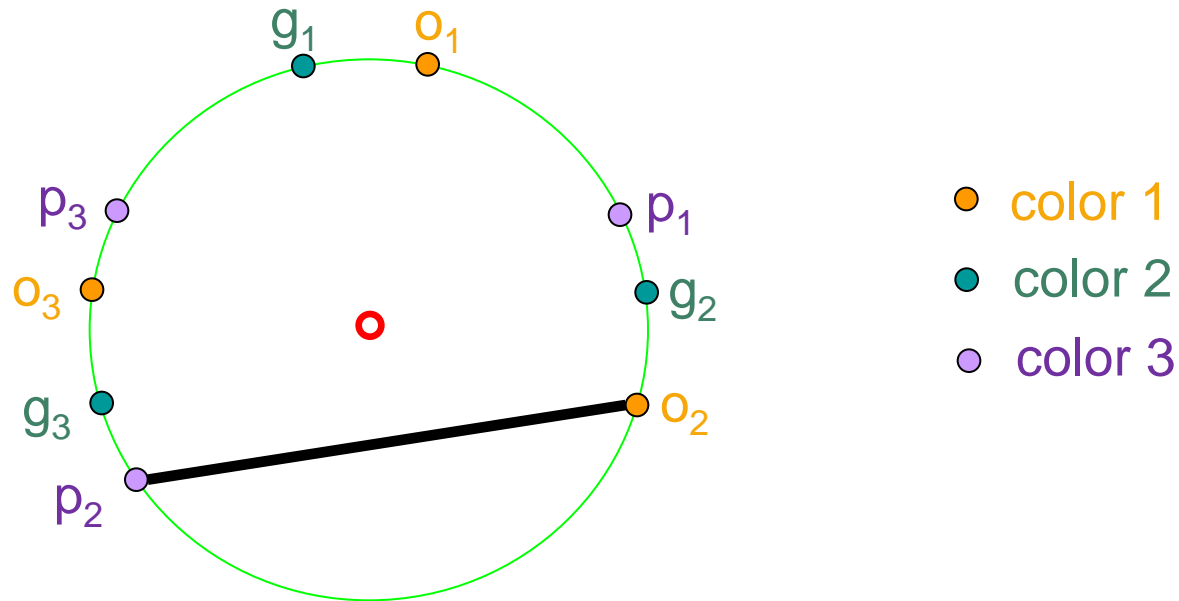
$$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$$

$S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$



# Transversal

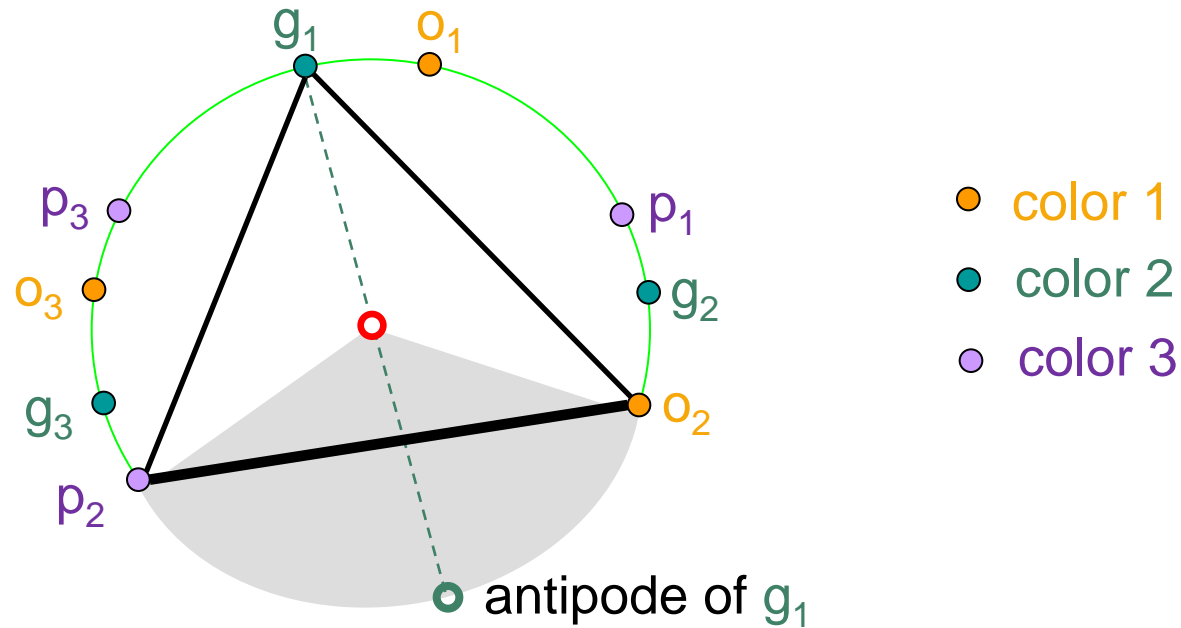
colourful set of  $d$  points (one colour missing)



$\hat{2}$ -transversal ( $o_2, p_2$ )

# Transversal

colourful set of  $d$  points (one colour missing)

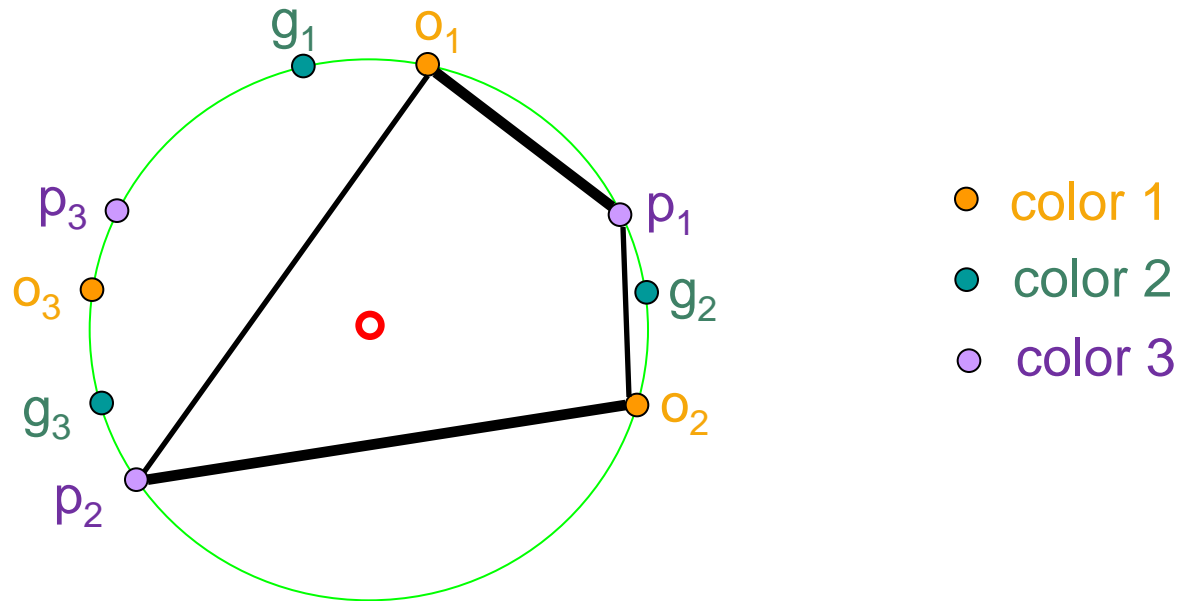


$\hat{2}$ -transversal  $(o_2, p_2)$  spans the antipode of  $g_1$

iff  $(o_2, p_2, g_1)$  is a colourful simplex

# Combinatorial (topological) Octahedra

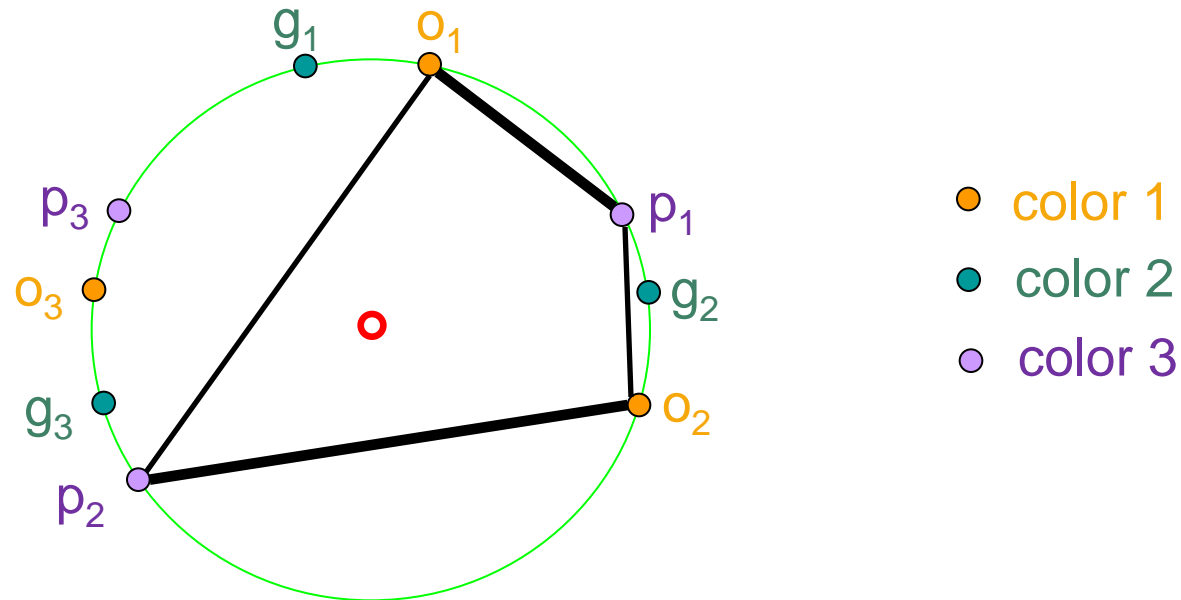
pair of disjoint  $\hat{i}$ -transversals



octahedron  $[(o_1, p_1), (o_2, p_2)]$

# Octahedron Lemma

origin-containing octahedra

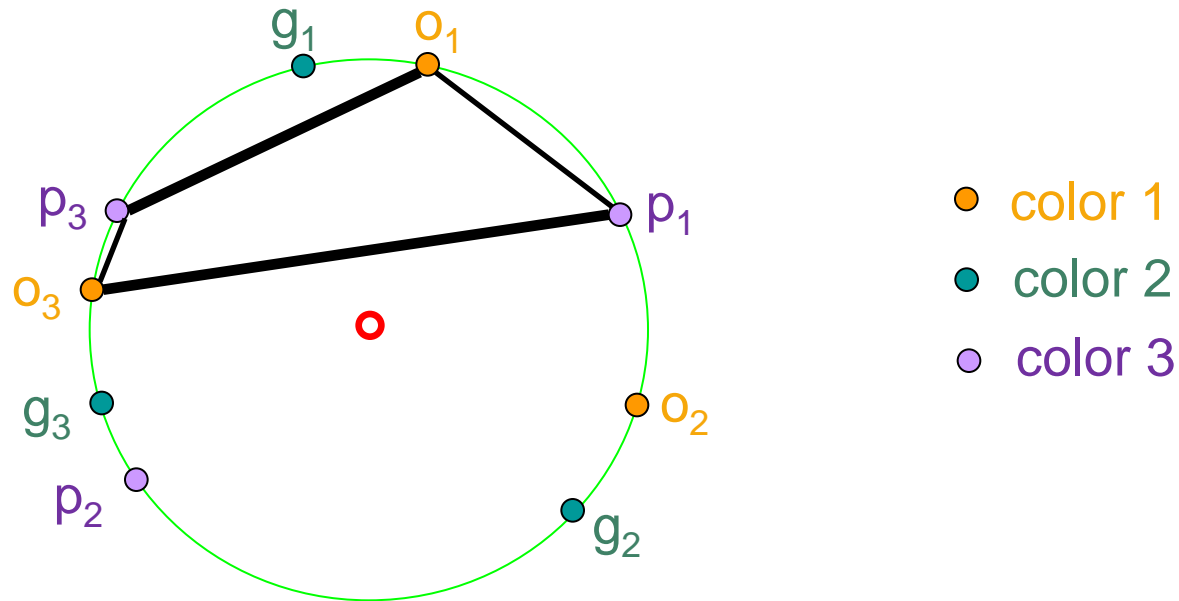


octahedron  $[(O_1, p_1), (O_2, p_2)]$

$2^d$  colourful faces span the whole sphere if it contains the origin (creating  $d+1$  colourful **simplexes**)

# Octahedron Lemma

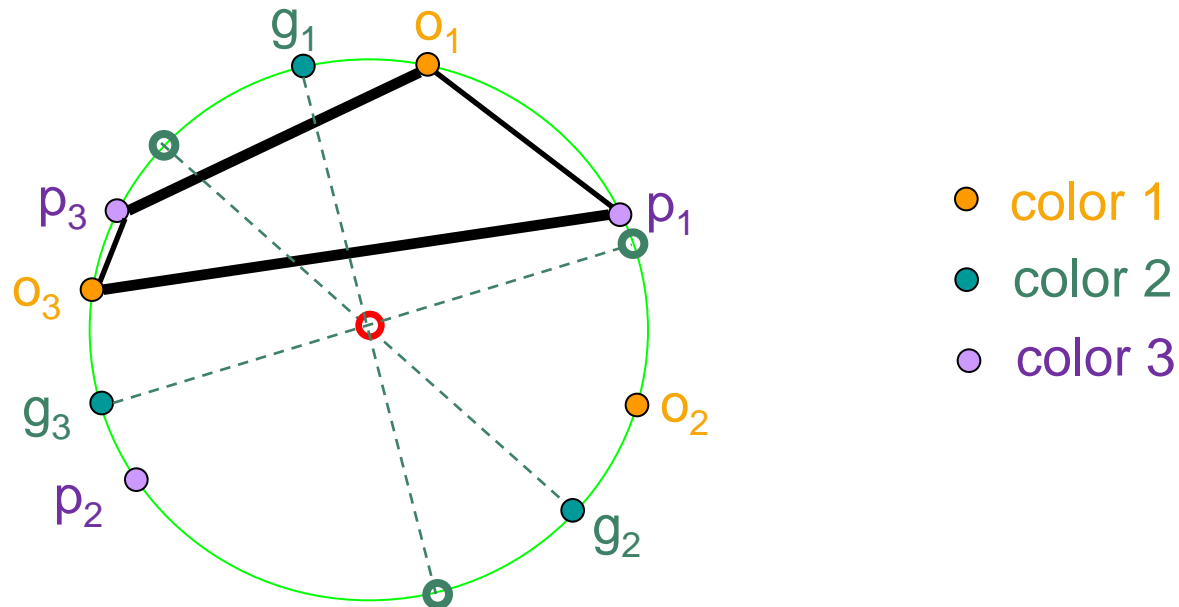
octahedron *not* containing the **origin**



octahedron  $[(o_1, p_3), (o_3, p_1)]$  does not contain  $p$

# Octahedron Lemma

octahedron *not* containing the origin

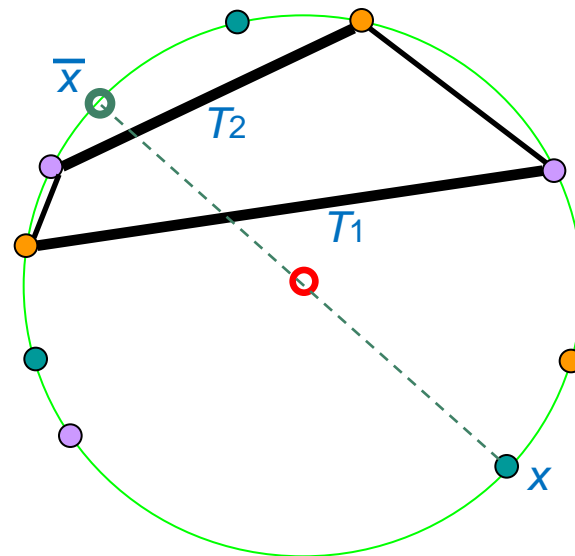
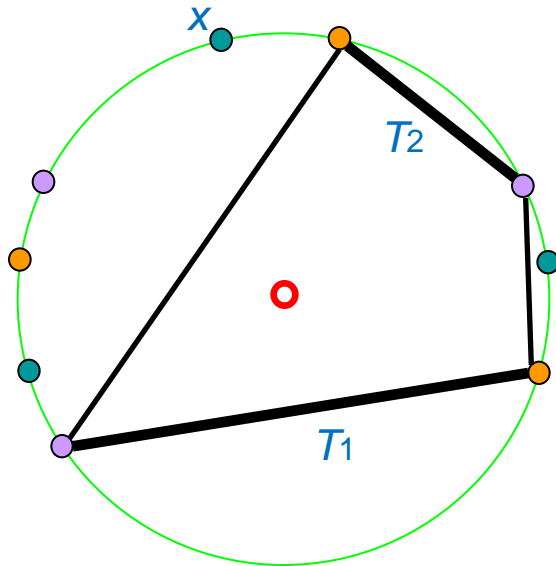


octahedron  $[(O_1, p_3), (O_3, p_1)]$  spans any antipode an even number of times

# Octahedron Lemma

Given 2 disjoint transversals  $T_1$  and  $T_2$ , and  $T_1$  spans  $\bar{x}$  (antipode of  $x$ ),

- either octahedron  $(T_1, T_2)$  contains  $p$ ,
- or there exists a transversal  $T \neq T_1$  consisting of points from  $T_1$  and  $T_2$  that spans  $\bar{x}$ .



# Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for  $\mu(d) = \min_{S, p} \text{depth}_S(p)$
- **Computational approaches** for  $\mu(d)$  for small  $d$ .
- Obtain an efficient algorithm to find a colourful simplex :  
*Colourful Linear Programming Feasibility* problem

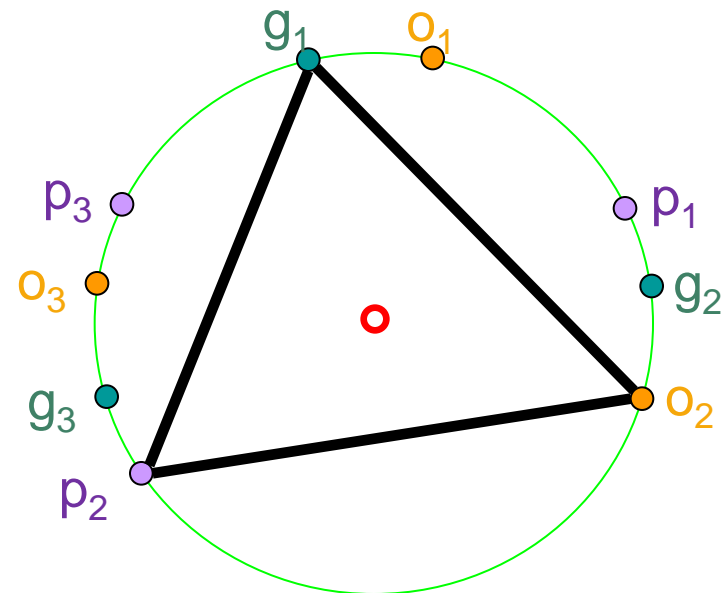
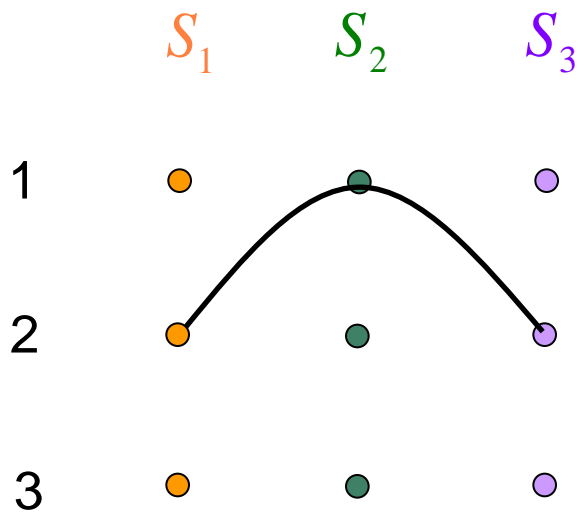
$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$



# Computational Approach

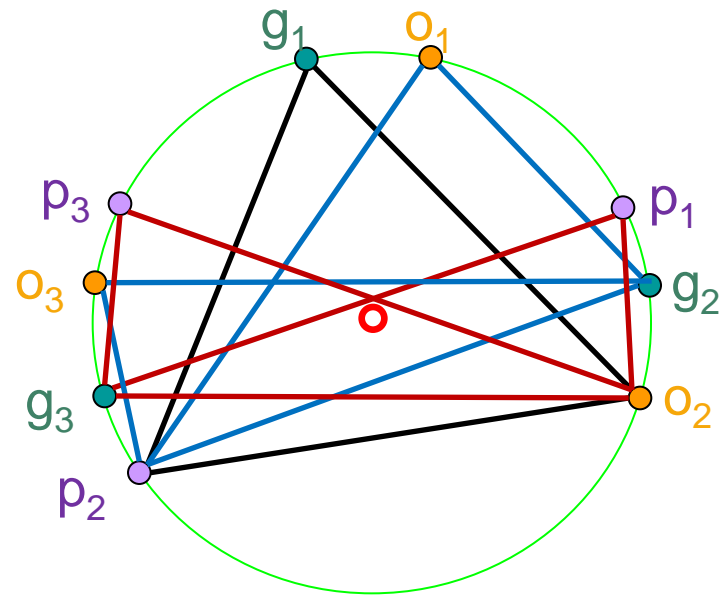
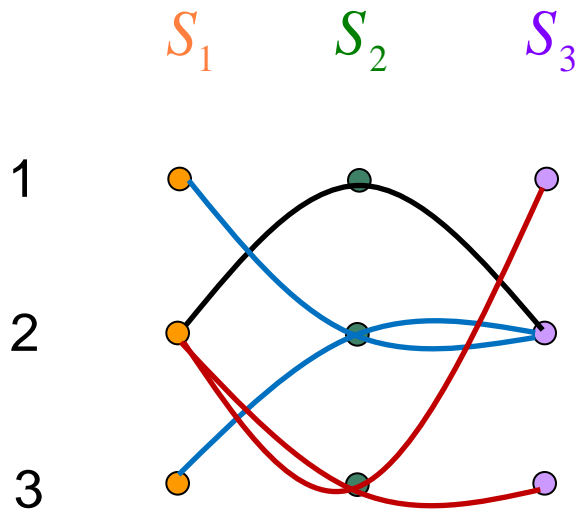
$(d+1)$ -uniform  $(d+1)$ -partite *hypergraph* representation  
of *colourful point configurations*



edge: colourful simplex containing  $p$

# Computational Approach

$(d+1)$ -uniform  $(d+1)$ -partite *hypergraph* representation of *colourful point configurations*



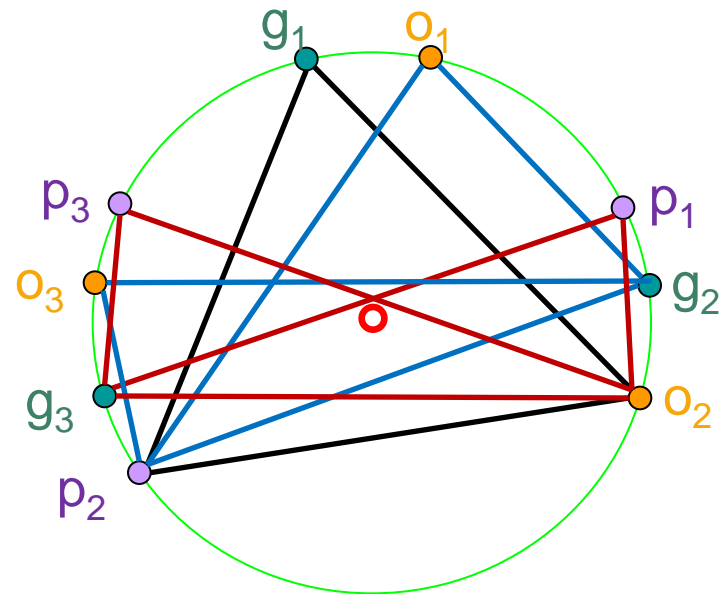
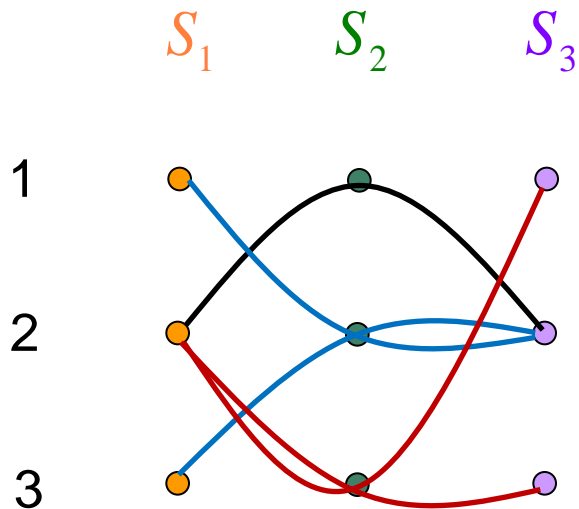
*necessary conditions:*

- **every** vertex belongs to at least 1 edge.
- **even** number of edges induced by subsets  $X_i$  of  $S_i$  of size 2

❖ reformulation of the *Octahedron Lemma*

# Computational Approach

$(d+1)$ -uniform  $(d+1)$ -partite *hypergraph* representation  
of *colourful point configurations*



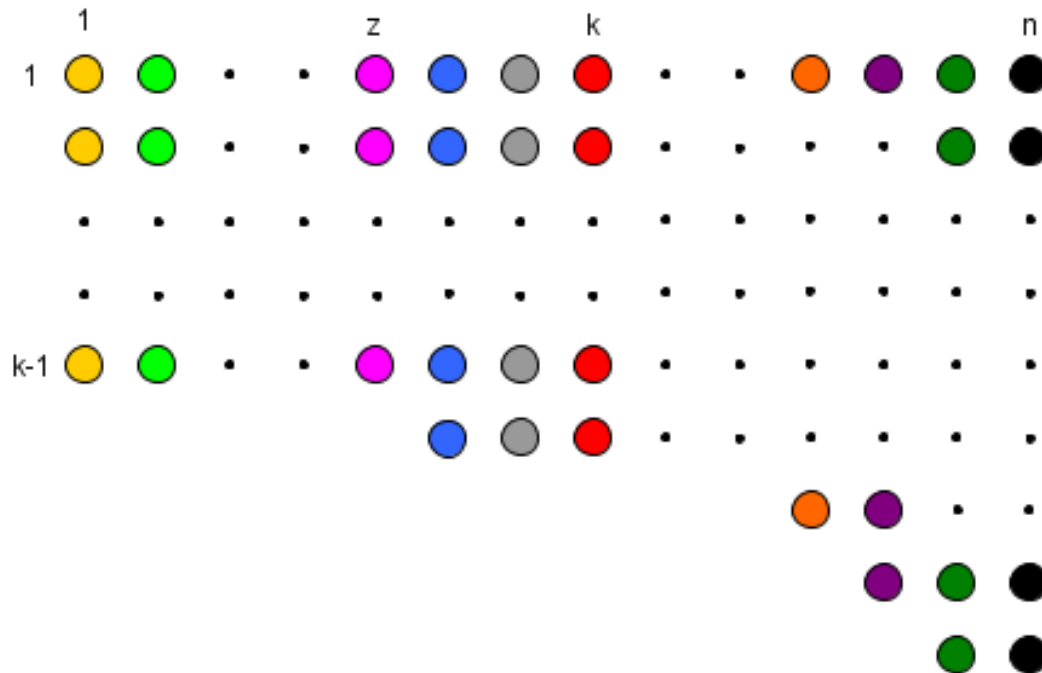
if no hypergraph with  $t$  or less hyper-edges satisfies the 2 necessary conditions, then  $\mu(d) > t$

$\Rightarrow$  computational proof that  $\mu(4) \geq 14$  [D., Stephen, Xie 2013]

❖ *isolated edge argument needed*

# Octahedral Systems

$n$ -uniform  $n$ -partite hypergraph  $(S_1, \dots, S_n, E)$  with  $|S_i| \geq 2$  such that the number of edges induced by subsets  $X_i$  of  $S_i$  of size 2 for  $i=1, \dots, n$  is even



# Octahedral Systems

- *even number of edges* if all  $|S_i|$  are even for  $i = 1, \dots, n$
- *symmetric difference* of 2 octahedral systems is octahedral
- existence of *non-realizable* octahedral system without isolated vertex
- *number* of octahedral systems:  $2^{\prod_1^n |S_i| - \prod_1^n (|S_i| - 1)}$

[D., Meunier, Sarrabezolles 2012]

# *Octahedral Systems*

- octahedral system without isolated vertex,  $|S_1| = \dots = |S_n| = m$   
has at least  $m(m+5)/2 - 11$  edges, implying:

$$\mu(d) \geq (d+1)(d+6)/2 - 11$$

- further analysis:  $\mu(4) = 17$

[D., Meunier, Sarrabezolles 2012]

# Colourful Research Directions

- Generalize the sufficient condition of Bárány for the existence of a colourful simplex
- Improve lower bound for  $\mu(d) = \min_{S, p} \text{depth}_S(p)$
- Computational approaches for  $\mu(d)$  for small  $d$ .
- **Obtain an efficient algorithm** to find a colourful simplex :  
*Colourful Linear Programming Feasibility* problem

[Bárány, Onn 1997] and [D., Huang, Stephen, Terlaky 2008]

$$\max_p \text{depth}_S(p) \geq \frac{\mu(d)}{(d+1)^{d+1}} \binom{n}{d+1} + O(n^d)$$

$p \in \text{conv}(S_1) \cap \text{conv}(S_2) \cap \dots \cap \text{conv}(S_{d+1})$   
 $S, p$  general position and  $|S_1|, |S_2|, \dots, |S_{d+1}| \geq d+1$

# Colourful Simplicial Depth Bounds

$$\mu(d) = \min_{S,p} \text{depth}_S(p)$$

$$\mu(1) = 2 \quad \mu(2) = 5 \quad \mu(3) = 10 \quad \mu(4) = 17$$

$$(d+1)(d+6)/2 - 11 \leq \mu(d) \leq d^2 + 1 \quad \text{for } d \geq 5$$

$\mu(d)$  even for odd  $d$

$$22 \leq \mu(5) \leq 26$$



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✓ *thank you*

# Tverberg Theorem

$n$  points can be partitioned into  $\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1$  colours, with a point  $p$  in convex hull intersection. [Tverberg 1966]

$\binom{\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1}{d+1}$  combinations to choose  $d+1$  colours.

If each combination has at least  $\mu$  colourful simplices. [Bárány 82]

$$\max_p \text{depth}_S(p) \geq \mu \binom{\left\lfloor \frac{n-1}{d+1} \right\rfloor + 1}{d+1} = \mu \binom{n}{d+1} + O(n^d)$$

$S, p$  general position