FIS - Statistical Machine Learning
Assignment 1

This homework is due May 7th (Tue.) 11:59 AM

You can either:

- Send your homework to marcocuturicameto+report@gmail.com. Please put the word report in the title of your email.
- Provide a handwritten copy. Please leave it in the mailbox of the course (in the Engineering Building 8) before Tuesday noon.

Positive Definite Matrices

A square $n \times n$ matrix $A$ is positive definite if

$$\forall x \in \mathbb{R}^d, x \neq 0 \Rightarrow x^T X x > 0.$$ 

Alternatively, $A$ is said to be positive semi-definite if

$$\forall x \in \mathbb{R}^d, x^T X x \geq 0.$$ 

1. Suppose $A$ is positive definite and symmetric. Prove that all the eigenvalues of $A$ are positive. What can you say of these eigenvalues if $A$ is a positive semi-definite matrix?

2. Prove that the sum of two symmetric positive definite matrices $A, B \in \mathbb{R}^{d \times d}$ is positive definite.

3. Prove that if $A$ is symmetric positive definite, then $\det A > 0$ and thus $A$ is invertible. On the contrary, show that if $\det A > 0$, then $A$ is not necessarily positive definite (you just need to provide a counterexample).

4. Prove that if $A$ is positive semidefinite and $\lambda > 0$, then $(A + \lambda I)$ is positive definite.

5. Prove that if $X \in \mathbb{R}^{d \times n}$ then $XX^T$ and $X^TX$ are both positive semidefinite.

6. Prove that if $X \in \mathbb{R}^{d \times n}$ has rank $d$, then $XX^T$ is positive definite (invertible).

7. Let $X \in \mathbb{R}^{d \times n}$ be a matrix, and $Y \in \mathbb{R}^n$. Prove that $\min_{\alpha \in \mathbb{R}^d} \|X^T \alpha - Y\|^2_2 + \lambda \|\alpha\|^2_2$ is attained for $\alpha = (XX^T + \lambda I)^{-1}XY$. 

1