Statistical Machine Learning
Assignment 1

This homework is due October 22th (Wed.) 11:59 PM

As you can see below, this homework only involves math questions.
You do not need to code this time.
Send your completed homework in pdf format to marcocuturicameto+report@gmail.com.
Please put the word report in the title of your email.

Positive Definite Matrices

A square $n \times n$ matrix $A$ is positive definite if
\[ \forall x \in \mathbb{R}^d, x \neq 0 \Rightarrow x^T X x > 0. \]

Alternatively, $A$ is said to be positive semi-definite if
\[ \forall x \in \mathbb{R}^d, x^T X x \geq 0. \]

1. Suppose $A$ is a symmetric matrix. What can you say about its eigenvalues?

2. Suppose $A$ is positive definite and symmetric. Prove that all the eigenvalues of $A$ are positive. What can you say of these eigenvalues if $A$ is a positive semi-definite matrix?

3. Prove that the sum of two symmetric positive definite matrices $A, B \in \mathbb{R}^{d \times d}$ is positive definite.

4. Prove that if $A$ is symmetric positive definite, then $\det A > 0$ and thus $A$ is invertible. On the contrary, show that if $\det A > 0$, then $A$ is not necessarily positive definite (you just need to provide a counterexample).

5. Prove that if $A$ is positive semi-definite and $\lambda > 0$, then $(A + \lambda I)$ is positive definite.

6. Prove that if $X \in \mathbb{R}^{d \times n}$ then $XX^T$ and $X^TX$ are both positive semidefinite.

7. Prove that if $X \in \mathbb{R}^{d \times n}$ has rank $d$, then $XX^T$ is positive definite (invertible).

8. Let $X \in \mathbb{R}^{d \times n}$ be a matrix, and $Y \in \mathbb{R}^n$. Prove that $\min_{\alpha \in \mathbb{R}^d} \|X^T \alpha - Y\|^2 + \lambda \|\alpha\|^2$ is attained for $\alpha = (XX^T + \lambda I)^{-1}XY$.

9. Compare this formula with the formula provided in Lecture 1. What is the advantage of introducing a positive $\lambda$ parameter in the optimization above?