Mean Reversion with a Variance Threshold

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Motivation
Assets
Assets
Mean Reverting Trade

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Mean Reverting Trade
Mean Reverting Trade
Mean Reverting Process

\[ x_t \in \mathbb{R}_+ \]

price of single asset at time \( t \)

- Mean-reversion = tendency to pull back to mean
- First-order Stationary processes are mean-reverting
Mean Reverting Process

\[ x_t \in \mathbb{R}_+ \]

price of \textbf{single} asset at time \( t \)

- Mean-reversion = tendency to pull back to mean
- First-order Stationary processes are mean-reverting

**Most financial assets are not mean reverting on a short time horizon**

in fact, most financial assets are \textbf{not} stationary
Purpose of Cointegration

\[ x_t = \begin{bmatrix} x_{1,t} \\ \vdots \\ x_{d,t} \end{bmatrix} \in \mathbb{R}_+^d, \text{ } d \text{ assets at time } t \]

- Cointegration: find \( y \in \mathbb{R}^d \) s.t. \( y^T x_t \) is stationary.
Purpose of Cointegration

\[ \mathbf{x}_t = \begin{bmatrix} x_{1,t} \\ \vdots \\ x_{d,t} \end{bmatrix} \in \mathbb{R}^d, \text{ } d \text{ assets at time } t \]

- Cointegration: find \( \mathbf{y} \in \mathbb{R}^d \) s.t. \( \mathbf{y}^T \mathbf{x}_t \) is \textbf{stationary}.

- Straightforward application to finance:

1. Estimate \( \mathbf{y} \) using historical data
2. trade basket \((\mathbf{y}^T \mathbf{x}_t)\) as a mean reverting asset.
Purpose of Cointegration

\[ d = 2: \text{Pairs trading.} \quad y = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \]
Estimate Cointegrated Relationships

- Existing work
  - Define a criterion to measure non-stationarity
    \[ \lambda(y) = \text{non-stationarity}(y^T x_t) \]
  - Minimize \( \lambda(y) \).
Estimate Cointegrated Relationships

- Existing work:
  - Define a criterion to measure non-stationarity
    \[ \lambda(y) = \text{non-stationarity}(y^T x_t) \]
  - Minimize \( \lambda(y) \).

- \( \lambda \) typically defined using time series modeling (VEC)
- Well studied topic in econometrics
Contribution

Stationarity is not enough
All Mean Reverting

The more variance $y^T x_t$ has, the better
All Mean Reverting

The faster $y^T x_t$ mean reverts, the better
Observation

Mean reverting assets that have

- small variance
- slow mean reversion

require more leverage to reach the same level of profit.
Observation

Mean reverting assets that have

- small variance
- slow mean reversion

require more leverage to reach the same level of profit.

Problem? cointegration techniques consider neither
Our Contribution

• Similar starting point:
  ○ Define criteria to measure mean-reversion

\[
\lambda(y) = \text{slow-mean-reversion}(y^T x_t)
\]

• Take into account the variance of \( y^T x_t \)
  ○ Minimize \( \lambda(y) \) subject to \( \text{var}(y^T x_t) \geq \nu \)
Criteria

We consider 3 new criteria

- **Portmanteau statistic**: minimize autocorrelations
- **Expected crossings**: maximize the expected number of mean crossings using crossing statistics
- **Predictability**: reuse older work by Box/Tiao ’77

Use semidefinite programming to optimize them
Portmanteau Criterion

- Portmanteau statistic of univariate process

\[ \text{por}_p(x) = \frac{1}{p} \sum_{i=1}^{p} \left( \frac{\mathbb{E}[x_t x_{t+i}]}{\mathbb{E}[x_t^2]} \right)^2 \]

- \( \propto \) Euclidean norm of autocorrellogram coefficients

- Used to test if a process is *white noise* (Ljung-Box).
Portmanteau Criterion

- If $x_t \in \mathbb{R}^d$, consider for $y \in \mathbb{R}^d$

$$\text{por}_p(y^T x_t) = \frac{1}{p} \sum_{i=1}^{p} \left( \frac{y^T A_i y}{y^T A_0 y} \right)^2,$$

where

$$A_i \overset{\text{def}}{=} \frac{1}{T - i - 1} \sum_{t=1}^{T-i} \tilde{x}_t \tilde{x}_t^T$$
Portmanteau Criterion

- Minimizing $\text{por}_p(y^T x_t)$ under variance constraint:

  \[
  \begin{align*}
  &\text{minimize} \quad \sum_{i=1}^p (y^T A_i y)^2 \\
  &\text{subject to} \quad y^T A_0 y \geq \nu \\
  &\quad \|y\|_2 = 1.
  \end{align*}
  \]
Portmanteau Criterion

- Minimizing $p \left( y^T x_t \right)$ under variance constraint:

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{p} (y^T A_i y)^2 \\
\text{subject to} & \quad y^T A_0 y \geq \nu \\
& \quad ||y||_2 = 1.
\end{align*}
\] (P)

- This problem is not convex
Semidefinite Relaxations

- Change of variables: \( Y = yy^T \)

- Trick: cast the problem as a **semidefinite program**

\[
\begin{align*}
\text{minimize} & \quad \sum_{i=1}^{p} \text{Tr}(A_iY)^2 \\
\text{subject to} & \quad \text{Tr}(A_0Y) \geq \nu \\
& \quad \text{Tr}(Y) = 1, \quad Y \succeq 0
\end{align*}
\]

- With the extra constraint \( \text{Rank}(Y) = 1 \), \((\text{SDP})\) and \((P)\) are **equivalent**.
S-lemma

• Brickman’61: when $d \geq 3$ and for two matrices $A, B$

$$\{(y^T Ay, y^T By) : y \in \mathbb{R}^d, \|y\|_2 = 1\} =$$

$$\{(\text{Tr}(AY), \text{Tr}(BY)) : Y \in \mathcal{S}_d, \text{Tr} Y = 1, Y \succeq 0\}$$
$S$-lemma

- Brickman’61: when $d \geq 3$ and for two matrices $A, B$

\[
\{(y^T Ay, y^T By) : y \in \mathbb{R}^n, \|y\|_2 = 1\} = \\
\{(\text{Tr}(AY), \text{Tr}(BY)) : Y \in S_n, \text{Tr} Y = 1, Y \succeq 0\}
\]

- When $p = 1$, the relaxation is exact: $Y^* \rightarrow y^*$

- When $p > 2$, one can find an approximate solution $\tilde{y}$ using $Y^*$ with suboptimality guarantees.
Experiments
Implied Volatility

Apple – AAPL Volatility Time Series

- Volatility data for 217 stocks over 8 years
- Clustered in 13 sectors
- Divided into 20 time windows, 85% train, 15% test
- Greedy selection of 50 baskets per window
Comparisons with Classic Cointegration

- Orthogonal Least Squares: smallest eigenvector of $A_0$
- Fully Modified OLS (Phillips’95)
- Johansen VEC model (Johansen’92)

Our criteria, with $\nu = 0.3 \times \text{median}(\text{var } x_i)$

- Portmanteau
- Crossing Stats
- Predictability
8 \times 10^{-4} \approx 20 \text{ Basis points}
Influence of Variance Threshold $\nu$

Trading Costs: 0 cents per contract (≈ 0 Basis Points)

Average Sharpe Ratio

Portm.  
BoxTiao  
Crossing

$\nu$ as a multiple of the median variance of all assets in basket