Computational Learning Theory
Formal Concept Analysis and Frequent Item Set Mining

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- Item Set Mining and the A Priori Algorithm
- Formal Concept Analysis
- Closed Patterns

ITEM SET MINING

A Simple Example

- Set of all items: $X = \{A, B, C, D, E, F\}$

Transaction ID	Item Sets
…	
3256	{A, C, D}
3257	{B, C, E}
3258	{A, B, C, E}
3259	{A, B, E, F}
…	….

- "Items A and C might be bought together."

Bit-vector Representation

- Every transaction can be represented as a bit-vector of n dimension, where $n = |X|$.

ID	A	B	C	D	E	F
…						
3256	1	0	1	1	0	0
3257	0	1	1	0	1	0
3258	1	1	1	0	1	0
3259	1	1	0	0	1	1
…						

Bag of Words

- Let $X = \{A_1, A_2, \ldots, A_k)$ be a finite set of words.
- For a sentence s, we define $T(s) = (x_1, x_2, \ldots, x_k)$ where

$$x_i = 1 \text{ if word } A_i \text{ appears in } s$$
$$= 0 \text{ o.w.}$$

for $i = 1, 2, \ldots, n$

Example

W =(arithmetic, book, compute, paper, suppose, square, symbol, write)

s_1: Computing is normally done by writing certain symbols on paper.

s_2: We may suppose this paper is divided into squares like a child's arithmetic book.

$$T(s_1) = (0, 0, 1, 1, 0, 0, 1, 1)$$
$$T(s_2) = (1, 1, 0, 1, 1, 1, 0, 0)$$

Mathematical Definitions

- Assuming a finite set of all items

$$X = \{A_1, A_2, \ldots, A_n\}$$

- A transaction is a pair $t = (i,\ T)$ of an identifier $i \in \mathbf{N}$ and a finite set of items $T \in X$
- A transaction database D is a finite set of transactions in which no pair of transactions have a same identifier, that is,

  $t = (i, T) \in D$ and $s = (j, S) \in D$ imply $i \neq j$.
- A pattern is a finite set of items.
  - Transactions are for training data patterns are rules.

Mathematical Definitions (2)

- For a pattern P and a transaction $t = (i, \ T\ )$, we say t satisfies P (or P matches t) iff $P \subset T$.
- Let $D(P) = \{\ t \mid P \text{ matches } t\ \}$.
- The support of P in a transaction database D is defined as $\text{supp}(P) = |\ D(P)\ |\ /\ |\ D\ |$.
  - The support is also called the relative frequency.

Definition of Learning Task

- Assuming a set of items X
- For a given transaction database D and a minimal support (threshold) σ s.t. $0 \leq \sigma \leq 1$, enumerate all patterns P s.t. supp(P) $\geq \sigma$.

A Very Simple Example

ID	A	B	C	D	E	F
1	1	0	1	1	0	0
2	0	1	1	0	1	0
3	1	1	1	0	1	0
4	1	1	0	0	1	1

supp({A})= supp({B})= supp({C})= supp({E})= 0.75,

supp({D})= supp({F})= 0.25

supp({A, B})=supp({A, C})=0.5, supp({A, D})=0.25,…

Monotonicity of the Support

Lemma For two patterns P and Q,

$$P \subseteq Q \Rightarrow \mathrm{supp}(P) \geq \mathrm{supp}(Q)$$

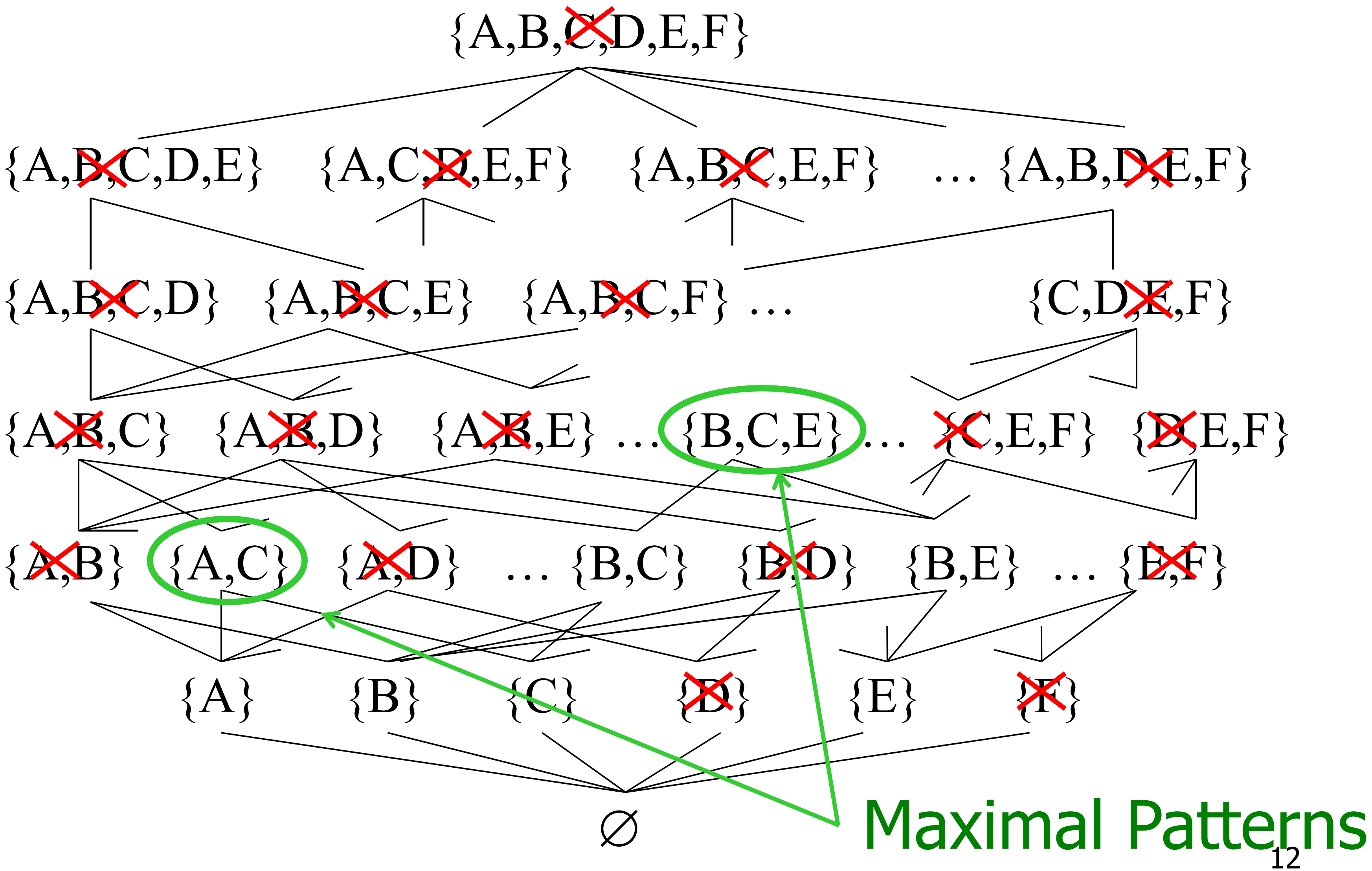
ID	A	B	C	D	E	F
1	1	0	1	1	0	0
2	0	1	1	0	1	0
3	1	1	1	0	1	0
4	1	1	0	0	1	1

supp({A})=0.75 ≥ supp({A , B})=0.25

supp({B})=0.5≥ supp({A , B})=0.25

supp({A})= 0.75 ≥ supp({A , C})=0.5

Maximal Patterns in the Hasse Diagram



FORMAL CONCEPT ANALYSIS

A Simple Example

- Set of all items: $X = \{A, B, C, D, E, F\}$

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…						

Context Table Representation

- Instead of "1", we use ●.

ID	A	B	C	D	E	F
…						
3256	●		●	●		
3257		●	●		●	
3258	●	●	●		●	
3259	●	●			●	●
…						

Formal Concepts

- A formal concept is a maximal rectangular filled with ●, without considering the ordering of law and column.

	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
g_1	●	●	●	●	●	●	●	●	●	●		
g_2	●	●	●	●			●	●				
g_3	●	●	●		●	●					●	●
g_4	●	●		●	●	●					●	●

	m_1	m_2	m_3	m_4	m_7	m_8	m_5	m_6	m_9	m_{10}	m_{11}	m_{12}
g_1	●	●	●	●	●	●	●	●	●	●		
g_2	●	●	●	●	●	●						
g_3	●	●	●				●	●			●	●
g_4	●	●		●			●	●			●	●

Intuitive Explanation

In the context of item set mining, a formal concept is a pair of a set A of transaction and a set B of items such that

- every transaction in A contains all items in B,
- every items in B is contained by all transactions in A,
- for every item i which is not in B, at least one transaction in A does not contain i, and
- for every transaction t which is not in A, at least one item is not contained by t.

Mathematical Definition

Errors Corrected

- A formal context K=(G, M, I) consists of two sets G (objects, *Gegenstand*) and M (attributes, *Merkmal*) and a binary relation $I \subseteq G \times M$.
- We define two functions $f : 2^G \to 2^M$ and $h : 2^M \to 2^G$

$$f(A) = \{\ m \in M \mid (g, m) \in I \text{ for all } g \in A\ \}$$

$$h(B) = \{\ g \in G \mid (g, m) \in I \text{ for all } m \in B\ \}$$

  - The pair (f, h) is called a Glois connection between 2^G and 2^M.
- A formal concept of K is a pair C=(A, B) with $A \subseteq G$ and $B \subseteq M$ such that $f(A)=B$ and $h(B) = A$, i.e.

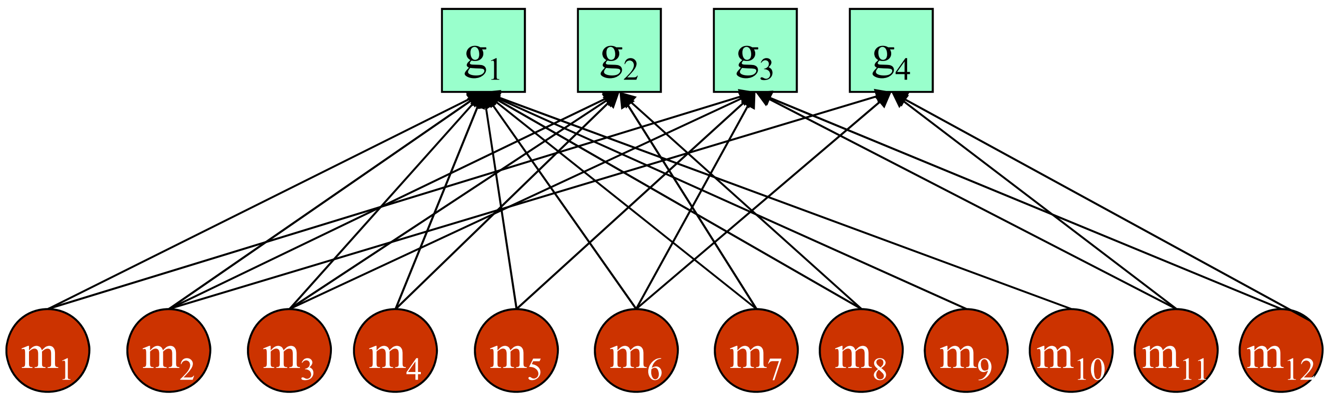
$$h(f(A))=A \text{ and } f(h(B))=B.$$

  - A is called the extent of C and B is called the intent of C.

Bipartite Graph Representation

- Every context table can be represented as a bipartite graph.
- Every formal concept is a represented as a bipartite clique.

	m_1	m_2	m_3	m_4	m_5	m_6	m_7	m_8	m_9	m_{10}	m_{11}	m_{12}
g_1	●	●	●	●	●	●	●	●	●	●		
g_2	●	●	●	●			●	●				
g_3	●	●	●		●	●					●	●
g_4	●	●		●	●	●					●	●



Some Propositions

For a context K=(G, M, I), A, A_1, $A_2 \subseteq G$ and B, B_1, $B_2 \subseteq M$,

- $A_1 \subseteq A_2 \Rightarrow f(A_2) \subseteq f(A_1)$
- $B_1 \subseteq B_2 \Rightarrow h(B_2) \subseteq h(B_1)$
- $A \subseteq h(f(A))$
- $B \subseteq f(h(B))$
- $A \subseteq h(B) \Leftrightarrow B \subseteq f(A) \Leftrightarrow A \times B \subseteq I$
- $h(f(A_1 \cup A_2)) = h(f((h(f(A_1)) \cup h(f(A_2))))$
- $f(h(B_1 \cup B_2)) = f(h((f(h(B_1)) \cup f(h(B_2))))$
- $A_1 \subseteq h(f(A_2) \Rightarrow h(f(A_1)=h(f(A_2))$

  and $h(f(A_1 \cup A)=h(f(A_2 \cup A))$
- $B_1 \subseteq f(h(B_2) \Rightarrow f(h(B_1)=f(h(B_2))$

  and $f(h(B_1 \cup B)=f(h(B_2 \cup B))$

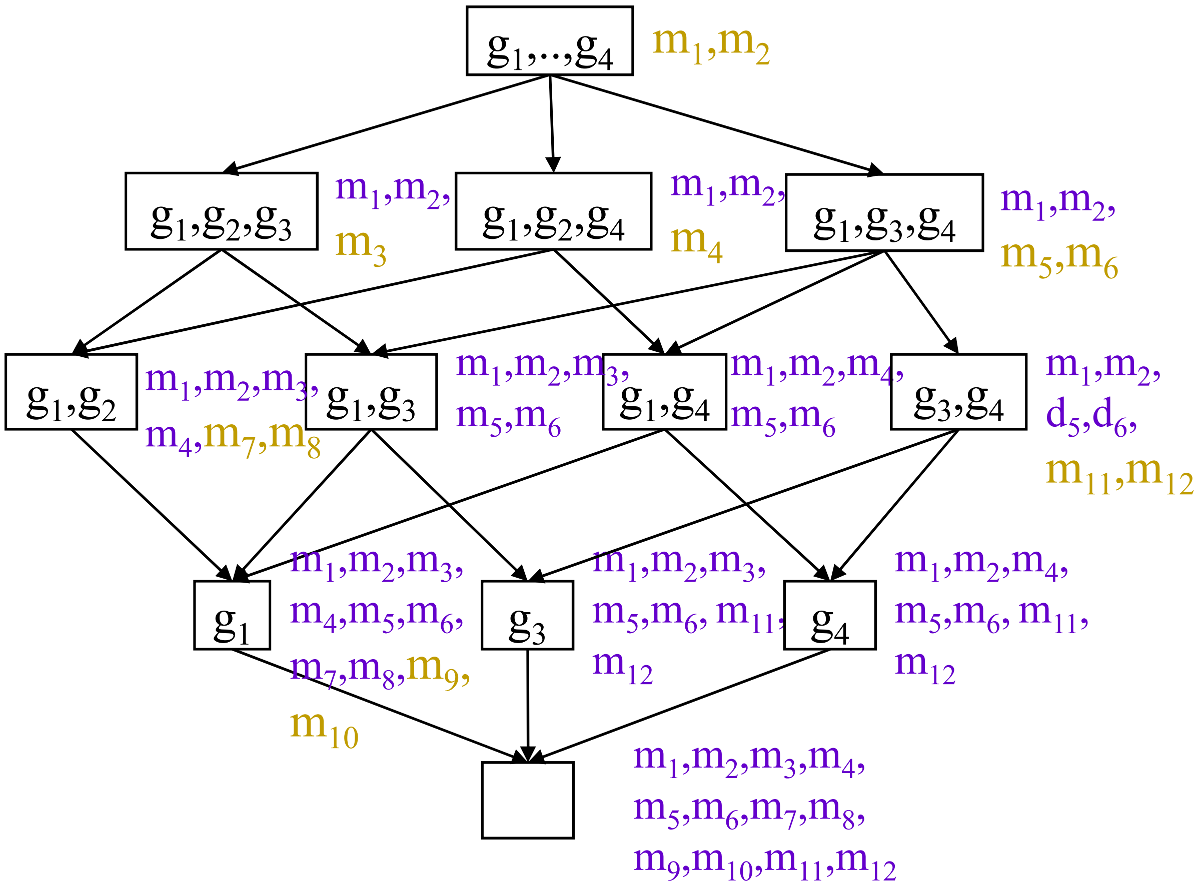
Some Propositions

For formal concepts C_1=(A_1, B_1) and C_2=(A_2, B_2),

$$A_1 \subseteq A_2 \Leftrightarrow B_2 \subseteq B_1$$

Hasse Diagram of FCs

- We can draw another Hasse diagram with all of the formal concepts.



CLOSED PATTERNS

Closed Item Sets [Pasquier et al.]

- For a transaction data, we let *G* is the set of all transaction id and *M* is the set of all items.
- An pattern *B* is closed iff $B = f(h(B))$, i.e, $(h(B), B)$ is a formal concept.

1	*a*		*c*	*d*	
2		*b*	*c*		*e*
3	*a*	*b*	*c*		*e*
4		*b*			*e*
5	*a*	*b*	*c*		*e*

$\sigma = 0.5$

Frequent closed pattern: *c*, *ac*, *be*, *bce*

Frequent but not closed pattern: *a*, *bc*, …

- For a transaction data, we let *G* is the set of all transaction ids and *M* is the set of all items.

Revised

Lemmas

Lemma For a context K=(G, M, I), $A \subseteq G$ and $B \subseteq M$

- $h(f(A)) = \cap_{g \in G} \{f(\{g\}) \mid A \subseteq f(\{g\})\}$
- $f(h(B)) = \cap_{m \in M} \{h(\{m\}) \mid B \subseteq f(\{m\})\}$

Corollary For closed patterns B_2, if $B_2 \subseteq B_1$ and $B_2 \neq B_1$, then supp(B_2) > supp(B_1).

Corollary For two closed patterns B_1 and B_2, if $B_2 \subseteq B_1$ and $B_2 \neq B_1$, then supp(B_2) > supp(B_1).

Lemma [Pasquier et al.] Every pattern B_1 of supp(B_1) = σ can be derived from some closed pattern B_2 of supp(B_2)= σ.

Additional Slide

Proposition

Proposition Every maximally frequent closed pattern is a frequent closed pattern.

An Example of Run(1)

ID	A	B	C	D	E	F
1	1	0	1	1	0	0
2	0	1	1	0	1	0
3	1	1	1	0	1	0
4	1	1	0	0	1	1

$\sigma = 0.5$

$C_1 = \{\{A\}, \{B\}, \ldots, \{F\}\}$

$L_1 = \{\{A\},\{B\},\{C\},\{E\}\}$

$C_2 = \{\{A, B\}, \{A, C\}, \{A, E\}, \{B, C\}, \{B, E\}, \{C, E\}\}$

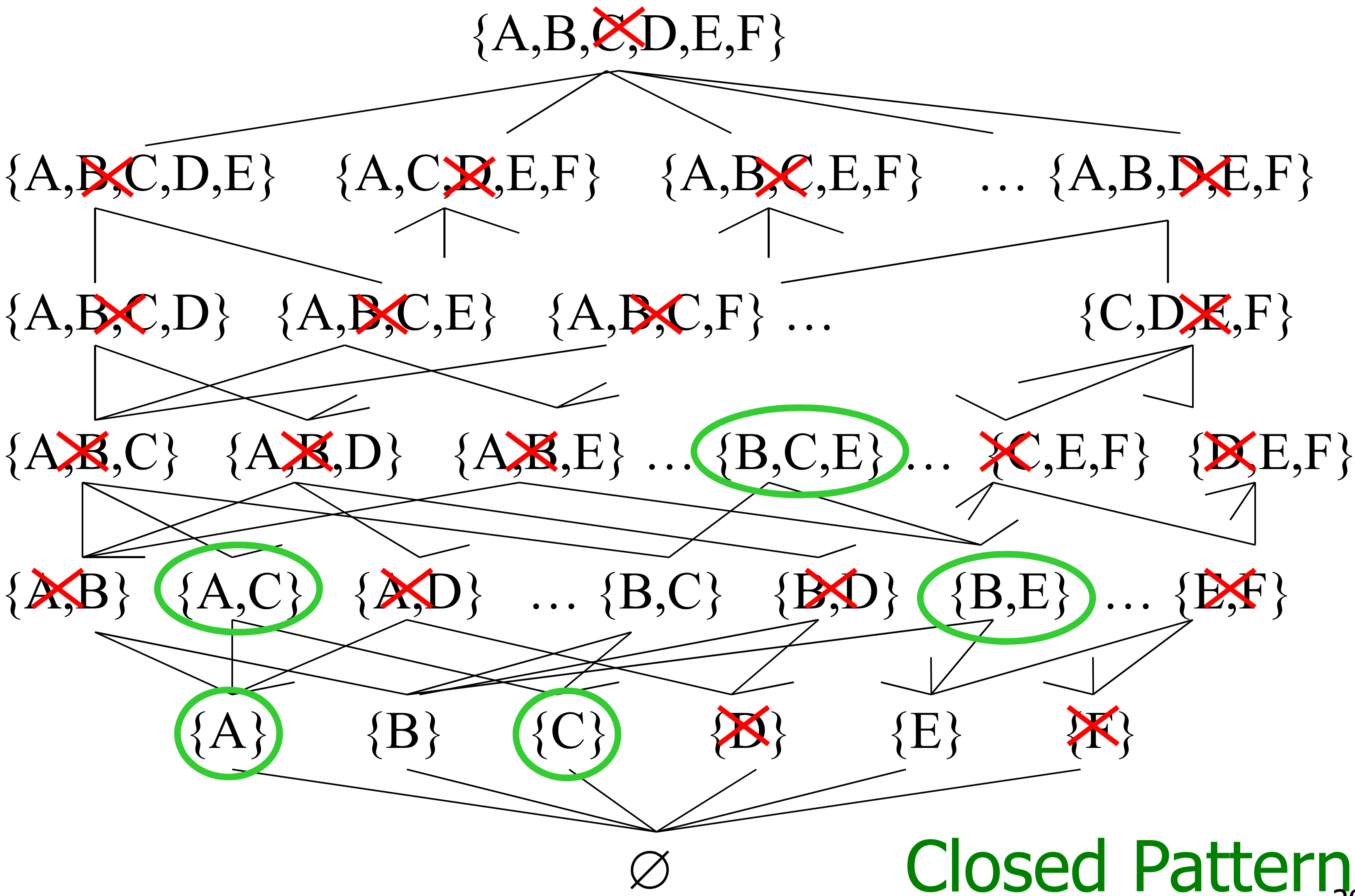
$L_2 = \{\{A, B\},\{A, C\}, \{A, E\}, \{B, C\}, \{B, E\}, \{C, E\}\}$

$C_3 = \{\{A,B,C\}, \{A,B,E\} \{B,C,E\}\}$

$L_3 = \{\{A,B,E\},\{B,C,E\}\}$

Errors Corrected

Maximal Patterns in the Hasse Diagram



Closed Patterns

Revised Additional Slide

Frequent Closed ItemSets

$\sigma = 0.5$

ID	A	B	C	D	E	F
1	●		●	●		
2		●	●		●	
3	●	●	●		●	
4	●	●			●	●

ID	A	B	C	D	E	F
1	●		●	●		
2		●	●		●	
3	●	●	●		●	
4	●	●			●	●

ID	A	B	C	D	E	F
1	●		●	●		
2		●	●		●	
3	●	●	●		●	
4	●	●			●	●

Revised Additional Slide

Frequent Closed ItemSets

$\sigma = 0.25$

ID	A	B	C	D	E	F
1	●		●	●		
2		●	●		●	
3	●	●	●		●	
4	●	●			●	●

ID	A	B	C	D	E	F
1	●		●	●		
2		●	●		●	
3	●	●	●		●	
4	●	●			●	●

ID	A	B	C	D	E	F
1	●		●	●		
2		●	●		●	
3	●	●	●		●	
4	●	●			●	●

ID	A	B	C	D	E	F
1	●		●	●		
2		●	●		●	
3	●	●	●		●	
4	●	●			●	●

Available Algorithm

Takeaki Uno and Tatsuya Asai, Hiroaki Arimura and Yuzo Uchida LCM: An Efficient Algorithm for Enumerating Frequent Closed Item, IEEE ICDM'04 Workshop FIMI'03